Imperial College London

Department of Electrical and Electronic Engineering

Final Year Project Report 2024

Project Title: **Robust Portfolio Optimization in Uncertain Markets**

Student: **Pranav Viswanathan**

CID: **01744658**

Course: **EIE3**

Project Supervisor: **Dr Imad Jaimoukha**

Second Marker: **Dr Adria Junyent-Ferre**

**Final Report Plagiarism Statement**

I affirm that I have submitted/will submit an electronic copy of my final year report to the provided EEE link.

I affirm that I have submitted/will submit an identical electronic copy of my final year report to the provided Blackboard module for plagiarism checking.

I affirm that I have provided explicit references for all the material in my Final Year Report that is not authored by me but represented as my own work.

I have not used linear learning models as an aid in the preparation of my report.

**Acknowledgements**

My journey through this final project, and by extension, my time at Imperial has been full of ups and downs, but extremely enlightening and rewarding. I’d like to thank my supervisor Dr. Imad Jaimoukha for not only his help ideating and navigating the direction of this project, but also teaching me several values for both learning and self-appreciation. I’d also like to thank my mother, father and little sister. Sindhu, Sajeev and Leya, this would not have been possible without you. Thank you for being a constant in my life.

Pranav

**Abstract**

Throughout the history of investing, Investment Managers have either made or broke fortunes in anticipation of certain stock investing opportunities being ludicrous. Warren Buffet immediately comes to mind with his Berkshire Hathaway now amassing an astonishing 892.60 Billion USD Market Cap [1]. Multitudes claim to use advanced portfolio selection and optimization techniques to maximize returns yet still, Investing and portfolio management is a very experience heavy field where confidence is inspired more by the vintage of an Investment Manager rather than the capabilities of the strategy he employs. Nevertheless, there are many statistical engineering concepts that can be applied to traditional investing strategies that can theoretically mitigate risk and increase returns when choosing a portfolio. With that thought in mind, this paper goes over the development and evaluates the performance of three investing frameworks revolving around changing parameters in Mean-Variance Optimization. These frameworks provide estimators derived through different ideologies: estimators derived from sample data, estimators derived from PCA denoised sampled data, and estimators derived from Bayesian Inferred sample data.

Contents

[**1.1 Introduction:** 6](#_Toc170132801)

[**1.2 Definitions:** 8](#_Toc170132802)

[**2.1.1 Mathematical Background: Markowitz Mean Variance Optimization** 9](#_Toc170132803)

[**2.1.2 Mathematical Background: Limitations of Markowitz Mean Variance Optimization:** 12](#_Toc170132804)

[**2.1.3 Mathematical Background: Efficient Frontier** 13](#_Toc170132805)

[**2.2.1 Mathematical Background: Bayesian Mean Variance Optimization** 15](#_Toc170132806)

[**2.2.2 Mathematical Background: Baye’s Formula** 16](#_Toc170132807)

[**2.2.3 Mathematical Background: Bayesian Inference** 17](#_Toc170132808)

[**2.2.4 Mathematical Background: Benefits of Bayesian Inference** 20](#_Toc170132809)

[**2.3.1 Principal Component Analysis** 21](#_Toc170132810)

[**2.3.2 Mathematical Background: Principal Component Analysis** 22](#_Toc170132811)

[**2.3.3 Mathematical Background: Benefits of PCA denoising:** 23](#_Toc170132812)

[**2.4: Mathematical Background: Combining Bayesian Inference and PCA** 24](#_Toc170132813)

[**3.1 Framework Development:** 25](#_Toc170132814)

[**3.2 Framework Development: onlyMarko.mlx** 26](#_Toc170132815)

[**4.2 Framework Development: onlyBayesian.mlx** 31](#_Toc170132816)

[**4.3 Framework Development: onlyPCA1.mlx** 32](#_Toc170132817)

[**4.4 Framework Development: BayesianPCA1.mlx** 34](#_Toc170132818)

[**4.1 Experiment Aims and Design:** 37](#_Toc170132819)

[**5.1 The Data Set** 38](#_Toc170132820)

[**6.1 Experiment 1 Aims and Design:** 41](#_Toc170132821)

[**6.2 Experiment 2 Aims and Design:** 42](#_Toc170132822)

[**7.2.1 Experiment 1 Results:** 43](#_Toc170132823)

[**7.2.2 Portfolio Value Tracking:** 44](#_Toc170132824)

[**7.2.3 Portfolio Value Tracking: BP2** 48](#_Toc170132825)

[**7.2.5 Portfolio Value Tracking: BP3** 52](#_Toc170132826)

[**7.2.6 Portfolio Value Tracking: Overall, BP1, BP2, BP3** 55](#_Toc170132827)

[**8.1.1 Experiment 2 Results:** 57](#_Toc170132828)

[**8.1.2 Markowitz Optimized weights for buy period ‘16-‘17** 57](#_Toc170132829)

[**8.1.3 PCA reduced weights for buy period ‘16-‘17** 58](#_Toc170132830)

[**6.1.4 Bayesian inferred weights for buy period ‘16-‘17** 63](#_Toc170132831)

[**8.1.5 PCA95-Bayesian inferred weights for buy period ‘16-‘17** 67](#_Toc170132832)

[**9.1 Evaluations and Conclusions:** 70](#_Toc170132833)

[**10 Further Work:** 71](#_Toc170132834)

[References 72](#_Toc170132836)

[**11 Appendix:** 74](#_Toc170132837)

[**A: Github Repo** 74](#_Toc170132838)

[**B: Mathematical Background: Analytical Solving of Minimization Problem** 74](#_Toc170132839)

[D: Different Types of Uncertainty that could be implemented: 76](#_Toc170132840)

[**C: Ethics, Legal and Safety Action** 77](#_Toc170132842)

## **1.1 Introduction:**

Public Stock Markets are heavily volatile and tumultuous, driven by public demand which is influenced by a variety of aspects, be it technological, economic or creative breakthroughs, or even international crises such as wars and health pandemics. Investment Managers are tasked with constantly finding the most optimal portfolio of assets to either maximize returns or minimize risk to appease stakeholders in Indexes. It is a strenuous job and requires constant data upkeep and monitoring.

Considering this, Harry Markowitz, widely regarded as the pioneer of portfolio optimization, put forth his Nobel prize winning paper ‘Portfolio Selection’ in 1952. In it, the mean-variance portfolio theory introduced by Markowitz remains influential in both research and practice. This theory provides a straightforward formula for the mean-variance efficient portfolio, which relies solely on two key population characteristics: the mean and the covariance matrix of asset returns. Ideally, when these parameters are known, investors can easily calculate the optimal portfolio weights according to their desired risk level or target return. However, in reality, the actual value of these parameters are unknown and subject to multiple conditions. Instead, investors use the sample means of assets and sample covariance matrices of the same as substitutes, creating what is known as the "plug-in" portfolio. This method is supported by classical statistics since the plug-in portfolio is a maximum likelihood estimate (MLE) of the optimal portfolio. Nevertheless, as highlighted by Michaud in 1989 and others, the plug-in portfolio performs poorly out-of-sample. The situation becomes even more problematic as the number of assets increases. This issue, referred to as the "Markowitz Optimization Enigma" by Michaud, has been further explored by researchers like Best and Grauer (1991), Green and Hollifield (1992), Chopra and Ziemba (1993), Britten-Jones (1999), Kan and Zhou (2007), and Basak et al. (2009), who document the challenges of constructing the mean-variance efficient portfolio using sample estimates.

The aim of this Final Year Project is to identify and implement various data manipulation techniques to achieve more representative estimators that will, to some extent, alleviate concerns of uncertainty and robustness. An additional aim of the experiment is to maximise returns and minimize risk on a created, controlled set of stock price data of companies from the Dow-Jones Index from the 1st of January 2010 to the 1st of January 2020.

This is a very well documented subject, with over nine hundred research papers being published on this topic and this project will evaluate how changing certain parameters in a robust portfolio optimisation algorithm affect returns and risk in a well-documented financial case study. This project is investigative and experimental in its approach to development, and should yield a greater understanding of how Investment Managers can utilize engineering concepts to improve estimators regarding risk and returns and maximise their profitability in uncertain markets considering the mean-variance framework.

In order to achieve this, three statistical frameworks were developed. The Markowitz Mean-Variance Optimisation, Principal Component Analysis mean-variance optimization, Bayesian Mean-Variance Optimization and by extension, a combination of both PCA and Bayesian Updating. The Markowitz model would be considered the basis and baseline for the experimental evaluation, whilst Principal Component Analysis (PCA) and Bayesian Inference would be supplemental to data fed to the Mean Variance Optimization Model and evaluated.

This report will provide an overview and analysis of the data set used, a mathematical explanation of concepts used, and how they might affect the Mean Variance Optimization model. Also presented is how these mathematical concepts were translated in code to suit the experiment design and relevance of the data set and be easily modified for use for any investor looking to generate high performing tangential portfolios. This report will also present the experiment design and an evaluation on both the experiment’s design and the models performance. Conclusions will be presented at the end and rationalized based on the experiment’s results. Furthermore, avenues for further development will be identified and built upon.

All frameworks were designed and run on MATLAB. Microsoft Excel was used for the evaluation-based data analysis. Appendix A holds a link to the code and experiment results used in the project.

## **1.2 Definitions:**

Considering the scope of the project, it would be beneficial to properly define and elaborate on key words related to the project title:

**Markets:** [5]. These are forums that trade forex, stock (both publicly and privately traded) or bonds.

**Portfolio:** A (financial) portfolio refers to a collection of assets an individual/corporate entity owns for the purpose of wealth management and generation over a period and/or maintenance of a trading dossier. [6] In this project, Assets will refer to a publicly traded stock of the financial managers choosing.

**Robust:** Seeing as public markets are extremely volatile due to constantly being traded and influenced by foreign factors such as world economies and technological/political breakthroughs, a financial portfolio generation model will always have to be adaptable, that is, perform effectively under constant change of variables and assumptions.

**Returns:** Any net gain or loss on an investment in an asset over a period.

**Risk:** Chance that an outcome or investment’s gain differs from an expected return. In the context of the project, and in line with Markowitz’s theory, it will be considered the variance of an asset, or the square root of the covariance of a portfolio.

**Uncertain:** As indicated above, when an expected return is indicated on a specific asset, it is essentially an estimate based on historical trends which a financial manager can never really know whether it will hold true for periods beyond historical data as the market as a whole is affected by many qualitative and random factors. Thus, the historical data estimates must be manipulated to represent more recent trends in the historical data.

The next section goes over all Mathematical and Statistical concepts used to develop the framework and derive expected return and risk estimators.

## **2.1.1 Mathematical Background: Markowitz Mean Variance Optimization**

Markowitz in his Nobel Prize winning paper ‘Portfolio Selection’ developed a mean variance framework that took a portfolio target return, portfolio target risk, asset expected returns (a sample estimator) and a sample covariance matrix (representing risk of the assets across sample data) both representing estimators grounded in historical data to achieve portfolio weights that would return the above targets. This section will go over how Markowitz’s portfolio optimization works.

To begin with, his standard mean-variance optimisation framework should be considered as a base. For a portfolio consisting of assets, a matrix is defined which corresponds to the budget weightage split across the assets (in the project’s case, a distribution of an investment manager’s budget across the assets.)

(1)

This scenario assumes that the investor cannot borrow money, or short sell. Alongside the weightage matrix, a matrix is retrieved, representing the expected (or estimated) returns of all the assets across a time period:

(2)

From this, a basic return maximisation problem can be constructed:

Where represents the expected return of the portfolio with asset weights . is a (nx1) matrix representing ones. This ensures the budget is maintained, and bars short selling by ensuring remains between 0 and 1 inclusive. This maximisation problem however does not consider the associated volatility of the portfolio assets. An investment manager would ideally aim to maximise the return whilst minimising the volatility of his portfolio. To incorporate this, Markowitz stated that a matrix representing the covariances of the portfolio asset returns is retrieved. This takes the form of symmetric, positive, semi-definite covariance matrix:

(3)

And subsequently, the risk is defined for an individual portfolio with asset weights as:

From this, a minimization program can be created:

Considering the maximisation and minimisation problem, the two can be combined to create a program that aims to attain a return whilst minimising the risk of the portfolio

And conversely, to obtain a risk whilst maximizing the return:

The two minimization and maximization problems could even be combined to create a minmax ‘cost’ function:

(4)

Where (the risk aversion factor) represents a scaling factor for the importance of risk whilst attaining the maximum return for a given level of risk. Adjusting gives the maximum return at a considered risk position (dependent on .)

Going forward, these minmax optimization problems will be considered as the Mean-Variance Framework Problems:

Solving these problems can be done via various ways:

1. **Monte-Carlo Simulation: the Monte-Carlo Model is very computationally heavy and rather than mathematically attempt to find the ‘optimal’ portfolio, the model runs through all possible solutions of the weightage matrix and outputs returns based on the historical returns of each of the assets stocks prices. The steps are as follows** [7]**:**
   1. **Identify the input variables. In this case, it is the weightage of the n assets in the portfolio. This is the variable that the Monte Carlo Model will vary throughout its repeated runs.**
   2. **Specify probability distributions of the independent variables. Use historical data to define a range of estimated returns and assign a probability distribution to the expected return for each asset.**
   3. **Using a random number generator (still subject to the weight and budget conditions,) generate random weights for each asset and then predict based on probability distribution of said asset the return obtained after a defined period. This is done multiple times to obtain a representative cumulative probability distribution for all assets, presenting multiple cases of portfolio weights and their respective global returns to the manager.**

**The Monte-Carlo Simulation is extremely computationally intensive, and only becomes more representative of how the weights of each asset in a portfolio would affect the overall return and risk with more and more runs of the simulation. Furthermore, since the probability distribution of the expected return must be calculated for each asset based on historical data available, it is still very tedious for a financial manager or computationally expensive for a system.**

1. **Analytical Approach**: with the help of Lagrange Multipliers, the minimization problem can be differentiated down to partial differentiations of the Lagrange constants in the minmax problem. These are then set to 0 (equations now known as optimality conditions) and solved to find values for each Lagrange Multiplier. This is not in the scope of the paper, but still explored in Section B of the Appendix.

Returning to equations (1). The paper defines how the expected returns vector and the estimated covariance matrix are calculated. Markowitz [5] defined the return of an asset across unit time i to be:

Where represents the stock price (close price in the project’s case) of the asset at time . This is done for all time periods for each of the stock prices:

Thus, the expected return estimator for each asset according to Markowitz:

This estimator is purely based on historical sample data of an asset, which is often not very representative of future trends on the return of the asset.

This is then collated in to a vector representing expected returns for all assets:

Returning to equation (3) The estimator for risk is considered to be the Covariance of the Returns Matrix:

Where:

Resulting in:

Where is the total number of assets. This is used as the risk estimator for subsequent Markowitz optimizations. Once again. The risk estimator is based solely on historical trends. There are many issues related to using the expected returns and covariance based on historical sample data, and this is explored below in section 2.1.2.

## **2.1.2 Mathematical Background: Limitations of Markowitz Mean Variance Optimization:**

While Markowitz’s portfolio optimization strategy formed the framework to modern portfolio strategies, There are several limitations associated with it:

**Assumption of Normally Distributed Returns**: MVO assumes that asset returns are normally distributed. In reality, asset returns often exhibit skewness and kurtosis, meaning they have fat tails and asymmetry that the normal distribution does not capture.

**Sensitivity to Input Estimates**: The optimal portfolio is highly sensitive to the estimates of expected returns, variances, and covariances. Small errors or changes in these inputs can lead to significantly different portfolios, making the optimization unstable and potentially leading to suboptimal investment decisions. The process of estimating the expected returns, variances, and covariances involves a degree of uncertainty and potential error (estimation risk). This can affect the reliability of the optimized portfolio. Furthermore the estimates for the expected returns and the risk are based purely on historical data which is only representative of historical market trends. Although it is nearly impossible to accurately predict future trends, Markowitz Mean Variance optimization essentially produces an averaged historical trend, which more often than not, is hardly indicative of where an asset will be at some point in the future.

**Static Model**: MVO is a single-period model and does not take into account the dynamic nature of markets and investments over time. It does not consider changing economic conditions, market trends, or the investor's changing circumstances and preferences.

Considering this, the project evaluates how PCA can make the system more robust, and how Bayesian inference alleviates some uncertainty and provide a more realistic estimate with more importance given to more recent historical trends.

## **2.1.3 Mathematical Background: Efficient Frontier**

Returning to the minimum risk optimization problem mentioned in Section 3.1.1:

Markowitz explained that for every expected portfolio return, there was a minimum variance that could be achieved. Thus, by varying the desired expected return, the minimum variance achievable could be solved for, and logged and plotted in an efficient frontier plot:

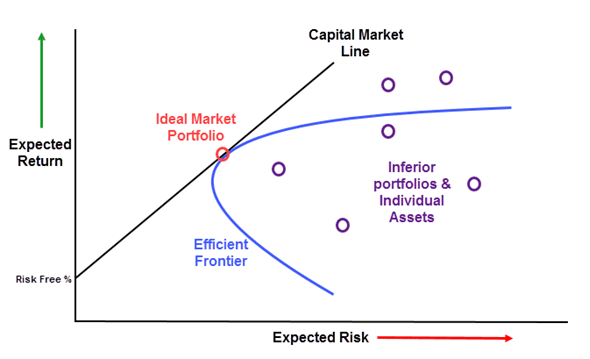


Figure 1: Efficient frontier example [5]

The Efficient frontier line (in blue) in figure 1 represents “efficient” portfolios that offer a minimum risk for an expected return. The ideal market portfolio, or the tangential portfolio, refers to the portfolio that offers the highest Sharpe Ratio. Also note that the Efficient Frontier is typically plotted with the expected return as the y axis and standard deviation representing risk as the x axis. The standard deviation for a particular portfolio is:

The Sharpe ratio is defined as below:

**Where**  refers to the expected return of the portfolio and refers to the risk-free rate of a risk free asset. A risk-free asset is defined as an investment with a guaranteed future return and negligible risk of loss. U.S. Department of the Treasury debt obligations, including bonds, notes, and particularly Treasury bills, are considered risk-free due to the backing of the U.S. government. Consequently, the yield on risk-free assets closely aligns with prevailing interest rates, reflecting their high safety and reliability. [6] However, since it is essentially a constant, for the purpose of this paper it is omitted:

This updated Sharpe ratio will represent the basis on which an investor will choose his portfolio. Typically, investors are of risk averse or return oriented nature. In order to appeal to both profiles, only the portfolio with the highest Sharpe ratio will be considered. This portfolio, known as the ideal market portfolio, offers the most return per unit risk to the investor and so should represent the needs of a risk-adjusted returns-oriented investor. The next section evaluates how Bayesian Inference can be used to more accurately predict historical trends to alleviate uncertainty concerns in returns/risk.

## **2.2.1 Mathematical Background: Bayesian Mean Variance Optimization**

Bayesian inference is a method of statistical inference that combines prior knowledge with observed data to update the probability estimates for a prior distrubution. This approach is grounded in Bayes' theorem, which relates current evidence to prior beliefs to form an updated belief.

This technique could be very beneficial to the mean-variance framework. Recall that return and risk estimators are only averaged returns and covariance matrices in Markowitz Mean Variance Optimization. As explained above, this often misrepresents historical trends as it may encompass events that are anomalous due to periods of economic misfortune not directly related to the assets stock price evolution. With Bayesian Inference, the project hopes that as new observed data becomes more available, the prior beliefs, which are the sample return and risk estimators used in Markowitz Mean Variance Optimization (recall that these are merely averaged sample historical data,) are updated or even skewed to more accurately incorporate trends in the new observed data. While this does not necessarily estimate future trends, it does skew the estimators in favor of more recent, and by extension, more relevant historical trends.

This section explores how the Covariance Matrix (risk estimator) and Expected Returns can be estimated based on the presence of new data to be more perceptive of how trends in both returns and risk have evolved given the new data. This should ideally provide less uncertainty in return/risk predictions.

## **2.2.2 Mathematical Background: Baye’s Formula**

To begin with, The Baye’s Formula is considered:

The formula states that the Probability of two events happening, , is the probability of A occurring multiplied by the probability of B given that A has occurred [7]:

However, it is also true for the probability of B multiplied by the probability of A given B occurs:

Therefore:

And so:

Or:

## **2.2.3 Mathematical Background: Bayesian Inference**

Considering the scope of the paper, Baye’s Theorem is rewritten as follows:

Where:

represents the probability density function of the prior mean and variance (standard Markowitz Estimators.)

represents the probability density function of the observed data mean and variance.

represents the probability density function of the posterior distribution estimate given the prior distribution and observed distribution that occurs.

represents a likelihood function of how probable it is to observe the data given a specific value. It quantifies the compatibility between the observed data and the parameter values.

Since does not depend on we can rewrite as:

Similarly, assuming has data following a multivariate distribution with mean and covariance: :

Yielding:

Where represents the dimension of .

As does not depend on , proportionality can be inferred and can be omitted:

Considering , it is asserted to be the maximum likelihood that the observed data follows an identical and independent distribution equivalent to :

Expanding using the probability density function for a multivariate normal distribution:

Similar to the process used for , proportionality is introduced:

Removing the product operator:

And since depends on neither nor , is omitted:

Now considering:

can be replaced considering the proportionality equations above:

Combining the exponentials:

Consider that , as n represents the number of samples observed. Thus:

Therefore, the exponential is updated:

Expanding the exponent, grouping terms dependent on and summing terms not dependent on as constants:

Let and and constants removed:

Which, to complete the square and represent a multivariate normal probability distribution function can be rewritten as:

Where the constants representing summed terms not dependent on A or B are removed:

Thus, the covariance of the posterior distribution =

And the mean of the posterior distribution =

Or:

Thus, the posterior distribution’s means and covariance have been inferred via the prior distribution and observed distribution to be used as explained in experimental procedure sections.

There is however, one assumption that this derivation makes that could skew results or affect the reputability of conclusions made at the end of the experimental process: All data, prior, observed and posterior are assumed to follow a multivariate normal distribution. Setting the prior and observed distributions as multivariate normal limits the risk definition to a single scenario, when another distribution may be better suited to model the returns. This may introduce estimation error and uncertainty once again to proposed portfolio weights obtained by mean variance optimization operations detailed above.

## **2.2.4 Mathematical Background: Benefits of Bayesian Inference**

Despite the potential risk of the assumption made above, utilizing Bayesian theory to infer a posterior distribution estimation has the following benefits:

**Incorporation of Prior Knowledge**: Bayesian inference allows for the integration of prior beliefs or knowledge into the analysis, which is beneficial when data is limited or historical data may not fully represent current conditions. In contrast, Markowitz optimization relies primarily on historical data without explicitly incorporating subjective beliefs or external information. For the projects purpose, the prior beliefs will be taken as the mean and variance of assets of the 5 year period preceding the new observed data.

**Flexibility with Small Samples**: Bayesian methods often yield more reliable estimates with smaller sample sizes compared to frequentist methods, such as Markowitz optimization. This advantage is particularly pronounced when informative priors are available or when data is sparse. Markowitz optimization can be less robust with smaller datasets due to its reliance on historical data quality and quantity.

**Quantification of Uncertainty**: Bayesian inference inherently quantifies uncertainty by providing a posterior distribution for the parameters of interest. This capability is crucial in financial decision-making to assess the range of potential outcomes and their associated probabilities. In contrast, Markowitz optimization typically provides a single optimal portfolio without directly addressing uncertainty unless additional methods or assumptions are applied.

This project will use Bayesian inference via the formulas derived above, and evaluate how portfolio generation was impacted, and how its subsequent performance on the dataset was affected.

## **2.3.1 Principal Component Analysis**

Principal Component Analysis (PCA) is a robust statistical machine-learning method applied in portfolio optimization to simplify financial data by reducing its dimensionality, uncover the key factors influencing asset returns, and enhance risk management.

This project leverages PCA on a returns matrix, in an attempt to reduce noise represented by un-trendlike changes in the returns of the asset. This should make the covariance matrices and expected returns between periods of asset returns be more resistant to small changes in the return matrix and more focused on the more important variations of returns. This would allow for underlying trends to be identified more accurately by the estimators for use in the Mean-Variance Framework.

The next section details how PCA is used to denoise a matrix:

## **2.3.2 Mathematical Background: Principal Component Analysis**

The following steps were used to denoise a matrix:

PCA is typically performed on standardized data. To achieve this, the mean of each variable (column) is subtracted from the dataset to centre it around the origin. Let be the original returns matrix with observations and variables. The standardized data matrix is computed as:

where is an vector of ones and is the mean vector of the columns of .

Then, the Covariance Matrix of the Standardized data is calculated:

Post which, Eigen Decomposition is performed to obtain the eigenvalues and eigenvectors of the return matrix:

Where is a matrix of ’s eigenvectors and is the diagonal matrix of eigenvalues. The diagonal is taken as it is asserted that there is no correlation between the eigenvectors.

From this, to retain a certain amount of variance explained by each eigenvector’s associated eigenvalue, the top eigenvectors that explain a cumulative variance threshold are retained:

matrix of eigenvectors explaining a certain threshold of variance.

Using this new matrix of eigenvectors, the standardized data matrix is projected to the reduced eigenvector space:

Where is he matrix of principal components representing a threshold value of total variance.

However, the mean variance framework used to optimize portfolios expects a matrix with dimensions equivalent to the returns matrix. Thus, the matrix is projected back to the original space using the retained eigenvectors:

Finally, the standardization is reversed by adding the mean vector back to the reconstructed data :

To summarize, the denoised data matrix is obtained by projecting the centered data onto the top principal components, reconstructing the data in the reduced space, and then adding back the mean vector. This process filters out noise by removing the components associated with smaller eigenvalues, which typically correspond to less significant patterns or noise in the data.

## **2.3.3 Mathematical Background: Benefits of PCA denoising:**

There are many theoretical advantages to denoising a returns matrix with PCA:

**Improved Risk Management:**

* Noise Reduction: PCA denoising helps in removing noise from the return series of assets, leading to more stable and robust covariance matrices. This reduction in noise allows for better estimation of risks associated with the assets.
* Stable Covariance Matrix: A more stable covariance matrix improves the reliability of portfolio optimization models, particularly those that are sensitive to the estimation of covariances, such as Mean-Variance Optimization (MVO).

**Enhanced Signal Extraction:**

* Focus on Principal Components: By concentrating on the most significant principal components, PCA denoising enhances the extraction of true underlying signals from the data. This helps in identifying the main factors driving asset returns, which can be critical for strategic asset allocation.
* Reduction of Spurious Correlations: PCA helps in filtering out random noise and spurious correlations that do not contribute to the true underlying structure of the data, leading to more accurate asset relationships and better decision-making.

## **2.4: Mathematical Background: Combining Bayesian Inference and PCA**

Via Bayesian Inference, the project hopes to generate estimators of the expected returns and covariance risk matrix that are more representative of more recent historical trends, as evidenced by the formulas derived in section 3.4.2. This should make the estimators more robust and alleviate uncertainty, as instead of an averaged sample estimator in the case of Markowitz estimators, Bayesian estimators are constantly learning and updating themselves based on the presence of new observed trends.

This can also be combined with PCA. As the project considers monthly returns over 10 years for 10 assets, there are exactly 1200 data points to consider. Some of these points however, may be anomalous and unrelated to driving factors under return trends. Using PCA and setting a threshold variance level, the anomalous reading can be filtered out. This should provide a more robust estimator for the risk and return as the filtered values are more akin to establishing a market trend.

The Bayesian inference can be split into two: the prior estimators and the observed estimators. From these estimators, a posterior distrubution is retrieved, and from it, the posterior estimators. In the experiment design below, the project considers the prior estimator to be akin to Markowitz estimators: A 5 year window before the observed data. The observed data will be represented by the year just before the buy period. Thus, in effect, the Bayesian inference also benefits from one additional year of data, as Markowitz estimators would only consider the 5 year period before the buy point.

This raises the question however, where can PCA be implemented? It was decided that PCA would be used to filter the prior returns matrix, and not the observed matrix as the observed matrix is only 10x10x12 data points long. Thresholding the variance there may result in key data points being lost, however, this also raises the concern as to whether without filtering, the observed data year with anomalous data points would greatly vary the posterior distrubution. This is something that is evaluated in the experimental results.

Also of note is the risk estimator takes the form of a 10x10 matrix. As the experiments use 4 different types of estimator generation, at any point, there would be four 10x10 covariance matrices to compare. This would be time and space consuming, so recalling the risk definition in the Mean-Variance framework:

It can be asserted that the risk consists of added up covariances and variances of each asset scaled by its weightage, and then square rooted. Thus, only the sum of rows of the covariance matrix will be considered, and called the cumulative variance and by extension, the cumulative risk of an asset.

The next section explores how the frameworks were created in MATLAB, and all their features and functionality.

## **3.1 Framework Development:**

In order to implement easy to use frameworks involving all the mathematical concepts above, MATLAB was used primarily due to it’s ability to quickly assess variables and it’s live output interface. All frameworks developed can be found in the Github link under Section A in the Appendix.

Four frameworks were developed:

* 1. **onlymarko.mlx:** This .mlx file takes an input of n x p stock prices and assets in the form of an excel sheet. Then based on a custom range set by the investor, quadprog is used to find portfolio weights based on the highest sharpe ratio formula mentioned in section 3.1.2. The expected returns, expected risk, portfolio weights and an efficient frontier are all outputted. The sample expected returns and sample covariance matrix are parameters that are calculated prior to quadprog’s execution.
  2. **onlyPCA1.mlx**: Performs the same functionality as onlymarko.mlx, however, PCA is used to denoise, to a certain threshold, the returns matrix generated from the n x p stock price/assets excel sheet. The parameters for sample returns and sample risk (covariance matrix) are all calculated from the denoised returns matrix.
  3. **onlyBayesian.mlx**: Via the formulas derived in section 3.4.2, a posterior estimate is formulated from a prior estimate that represents sample returns and risk from a custom range set by the investor, and observed returns and risk, also from a custom range set by the investor. Then, similar to onlymarko.mlx and onlyPCA1.mlx, portfolio weights are generated according to the highest attainable sharpe ratio along the efficient frontier, and the expected returns and expected risks are also outputted.
  4. **BayesianPCA1.mlx**: Via the formulas derived in section 3.4.2, a posterior estimate is formulated from a prior estimate that represents sample returns and risk from a custom range set by the investor, and observed returns and risk, also from a custom range set by the investor. However, the prior returns matrix is subjected to denoising, leading to potentially more accurate Prior estimators. Then, similar to onlymarko.mlx and onlyPCA1.mlx, portfolio weights are generated according to the highest attainable sharpe ratio along the efficient frontier, and the expected returns and expected risks are also outputted.

The next section will go over how each function was translated in code, including all setbacks and limitations.

## **3.2 Framework Development: onlyMarko.mlx**

The onlyMarko.mlx framework was designed to closely represent Markowitz’s portfolio theory. Given an excel sheet containing the stock prices of various assets through time, the framework would generate portfolio weights that yield the highest sharpe ratio based on expected returns and expected risk.

In order to read the excel file and store the stock price readings into a matrix for further use, MATLAB’s readmatrix() formula was used:

A close-up of a computer code

Description automatically generated

Figure 1: excel file import

Here, the investor would only have to change the variable filename to his excel sheets name. fulldata.xlsx was the data used for the testing of each framework, and comprised of asset stock prices detailed in the Data Set section before the experimental analysis. Additionally, to verify that the data going into cdata (matrix used to represent the stock prices) was complete, the framework outputs the dimensions which can be verified by the investor. In the report’s case, these dimensions were 120x10.

A computer code with text

Description automatically generated with medium confidence

Figure 2: return calculations

Post the file import, the returns were calculated as per the formula explained in section 3.1.1:

These returns are then collated into a returns1 matrix. After that, in the last line in the above snippet, the investor can set the bounds of historical returns to use by adjusting the matrix range definers, the first range 12:71 representing in the projects case, year 2011 to 2015. The second range can be used to vary the number of assets included in portfolio generation. For the projects case, this was untouched, and portfolio weights were generated for all 10 evaluated assets.

In order to find sample expected returns and sample risk, MATLAB’s mean() and cov() functions were used to define mu and cov\_matrix respectively:

A close up of a math equation

Description automatically generated with medium confidence

Figure 3: Parameter estimation, onlymarko.mlx

From there, in order to define a range of expected returns with which an efficient frontier can be plotted, the linspace() and min() and max() functions was used to create a vector of equidistant expected returns, that are arranged in an increasing order:

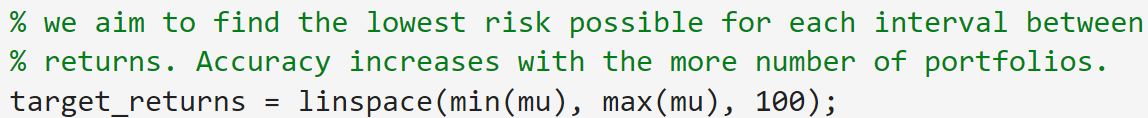


Figure 4: target returns; quadprog framework

Important to note here is that due to computational limitations, the max number of portfolios that could be generated to form an efficient frontier was 100. However, as the portfolios directly contribute to the creation of the efficient frontier, increasing the number of target\_returns will also increase the accuracy of the efficient frontier generation, and subsequently, the highest sharpe portfolio, which exists for one of the returns in target\_returns.

A black text on a white background

Description automatically generated

Figure 5: Array setup; quadprog framework

From there, arrays are set up to store the associated risks, returns, and asset weights for each portfolio eventually generated by the quadprog function:

A screenshot of a computer program

Description automatically generated

Figure 6: Quadprog Mean-Variance Optimizer

Above, the quadprog function was interatively called 100 times (the range of target returns defined by the linspace() function) in order to find portfolio weights corresponding to the lowest achievable risk for each target return. MATLAB’s quadprog function works as follows:

This was very similar to the minimization objective function described in section 3.1.2:

Thus, the following constants were defined:

**Objective Function**:

* H (covariance matrix of asset returns): cov\_matrix
* f (linear term, zero in this case): zeros(num\_assets, 1)

The objective function being minimized in quad prog is now:

​

which represents the portfolio variance (risk.)

**Equality Constraint**:

* Aeq (equality constraint coefficients): ones(1, num\_assets)
* beq (equality constraint bounds): 1

This ensures that the sum of the portfolio weights equals 1:

**Inequality Constraint**:

* A (inequality constraint coefficients): -mu'
* b (inequality constraint bounds): -target\_return

This ensures that the portfolio's expected return is at least the target return (in each interation of the for loop:)

**Bounds**:

* lb (lower bounds on weights): zeros(num\_assets, 1), if short-selling is not allowed, otherwise -1 \* ones(num\_assets, 1)
* ub (upper bounds on weights): ones(num\_assets, 1)

This ensures that the weights are within specified limits (e.g., no short selling means weights are non-negative). After Experiment 2 and understanding how the Bayesian inferred parameter estimators behaved, the shortselltoggle condition was added. If an investor desires, quadprog can be manipulated to find portfolio weights that can extend to negativity representing short positions on specific assets. The bounds currently are limited to bounds of [-1, 1] however this can be changed very simply:

If (shortselltoggle == 1)

* lb (lower bounds on weights): -(desired lower bound) \* ones(num\_assets, 1)
* ub (upper bounds on weights): (desired upper bound) \* ones(num\_assets, 1)

The code loops over each target\_return value, solves the QP problem using quadprog, and stores the results as follows:

* ‘x’ contains the portfolio weights that minimize risk for the given target return.
* ‘portfolio\_weights’ stores the weights for each target return.
* ‘portfolio\_returns’ stores the achieved returns (which should be close to or greater than the target return).
* ‘portfolio\_risks’ stores the portfolio risks (standard deviation of returns).

Also note that only If quadprog finds a solution (solutionindicator == 1), it stores the results; otherwise, it stores ‘NaN’ to indicate that no solution was found for that target return.

A screenshot of a computer code

Description automatically generated

Figure 7: index cleaning; quadprog

Subsequently, the ‘NaN’ values are removed from the respective portfolio arrays. At this point, the framework has everything required to plot an efficient frontier and find the high sharpe portfolio. In order to find the high sharpe portfolio, the return is divided by the risk of each portfolio and stored in sharpe\_ratios:



Figure 8: sharpe ratio calculation

The ‘rf’ variable refers to the risk free rate, which was taken as 0 during testing, but left in the event a financial manager wishes to incorporate a risk free asset.

Post the generation of the sharpe\_ratios matrix, the portfolio with the highest sharpe ratio is located:

A screenshot of a computer code

Description automatically generated

Figure 9: finding Sharpe High portfolio

And the weights, expected return, and risk are stored in ‘max\_sharpe\_weights’, ‘max\_sharpe\_returns’, ‘max\_sharpe\_risk’ variables.

Then, to plot the efficient frontier and output the high sharpe portfolio, MATLAB’s plot function was used to plot each target return portfolio on an xy plane representing standard deviation (risk) – expected return. Additionally, the tangential portfolio was also plotted in red to verify that results were correct; drawing of an efficient frontier and placement of the optimal portfolio:

A screenshot of a computer code

Description automatically generated

Figure 10: plot function for efficient frontier and Sharpe Portfolio

Finally, the max sharpe portfolio’s expected returns and risk where outputted, along with the weights. Below is an example of the output for the same range, 2011-2015. Please note that these outputs were mainly for internal testing and verification purposes.

A graph with a red dotted line

Description automatically generated

Figure 11: Example Output

## **4.2 Framework Development: onlyBayesian.mlx**

In order to incorporate the Bayesian inference of the posterior estimators taken from two separate sources, the formula from section 3.4.2 were replicated in Matlab code:

covariance of the posterior distribution =

mean of the posterior distribution =

Or:

In order to allow for more diverse applications, there was no logic applied connecting the two different sample estimators. Doing this allows for an investor to specify his own ranges for prior and observed returns data.

The entire framework was similar to onlymarko.mlx, only changing the input to quadprog. The new inputs were the posterior distrubution means and covariance matrix:

A screenshot of a computer program

Description automatically generated

Figure 12: Performing Bayesian inference

Where range0 and range1 correspond to the prior returns sample data and the observed returns data. These were taken from the returns matrix calculated the same way as in onlymarko.mlx:

A screen shot of a computer code

Description automatically generated

Figure 13: File import and returns split for Bayesian Inference

Post these steps, similar to onlymarko.mlx, Sigma\_star representing the covariance matrix of the new posterior risk estimator and mu\_star representing the new posterior returns estimator were fed to quadprog, and weights were retrieved in the same way as in onlymarko.mlx.

## **4.3 Framework Development: onlyPCA1.mlx**

To implement noise reduction as defined in Section 3.4.1, the MATLAB pca() function was used.

Once again, thanks to using the same quadprog framework, the only changes in the onlyPCA1.mlx file compared to onlymarko.mlx was the noise reduction to a cumulative variance level. This was done with the pca() function which conveniently performs eigendecomposition and returns the following:

coeff: Principal component coefficients (eigenvectors). Each column represents a principal component.

score: Principal component scores. The data projected onto the principal components.

latent: Eigenvalues of the covariance matrix of the data. Each value represents the variance explained by each principal component.

explained: Percentage of the total variance explained by each principal component.

Prior to this step, the returns matrix (corresponding to selected range as in onlymarko.mlx) has to be standardized. This is performed by subtracting the mean from each datapoint, and the dividing by the standard deviation:

A white background with black and green text

Description automatically generated

Figure 14: Data Standardization, PCA Mean Variance Framework

Then, the PCA function is called on ‘data\_standardized’:

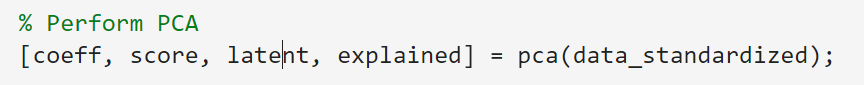


Figure 15: pca() function in PCA MVF

From here, the Cumulative variance ‘cumulative\_explained’ is calculated by the cumsum() function on explained, this value will be 100:



Figure 16: Listing cumulative variances of Principal Components

Post which, the number of principal components to be retained explaining a certain threshold is found using the find function. This value here is what the investor can vary for different levels of noise reduction. In the example, it is 95%:



Figure 17: Selecting relevent eigenvectors

Then, from ‘score’ which is already the data projected on each principal component, the relevant components are taken, representing the denoised data on principal component axes:



Figure 18: reducing data to relevant eigenvectors

And the ‘reduced\_data’ matrix is projected back on to the original axes of the return matrix by multiplying with the relevant eigenvectors from ‘coeff’:

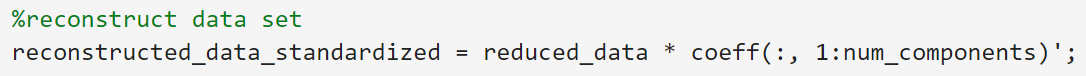


Figure 19: projecting dataset from PC axes to returns axes

Finally, to reverse the standardization performed initially, the ‘reconstructed\_data\_standardized’ matrix elements are multiplied by ‘data\_std’ and ‘data\_mean’ is added back to them:

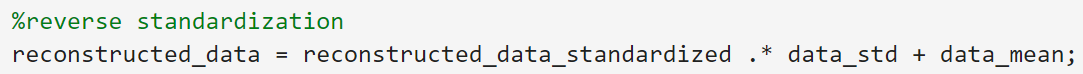


Figure 20: reversing standardization

From this point onwards, steps taken in onlymarko.mlx were replicated. The expected returns and risk were taken from the reconstructed\_data returns matrix via the mean() and cov() functions for use in the quadprog framework, which eventually outputs weights again for the highest sharpe ratio.

## **4.4 Framework Development: BayesianPCA1.mlx**

This section describes the development of the BayesianPCA framework. Recall from section 2.4, it was theorized that when PCA is applied to denoise the prior returns matrix, more accurate prior estimators would be derived which would aid in generation of posterior estimators via methods in 3.2 onlyBayesian.mlx. Though it was decided PCA would not be performed on the observed returns matrix, the framework was still implemented in the event the observed data takes the form of a much wider dimension:

A screenshot of a computer code

Description automatically generated

Figure 21: PCA on both ranges for Bayesian Inference

The above sequence follows the same procedure in onlyPCA1.mlx, only replicated for both range0 and range1 as reconstructed\_data0 and reconstructed\_data1 for the prior returns sample data and the observed returns sample respectively.

A screenshot of a computer code

Description automatically generated

Figure 22: Bayesian update in BayesianPCA.mlx

The above sequence details how the Bayesian update was performed similarly to onlyBayesian.mlx. Note that in this snippet, Sigma1 takes the covariance of the observed data only, following the principal in section 3.2.

However, upon taking the inverse of the Sigma0 covariance matrix, which was the Covariance of the prior reduced sample returns, MATLAB put forth the following issue:



Figure 23: error presented by MATLAB on inverse of SIgma1

Due to poor scaling of the reconstructed\_data0, which was the denoised prior returns matrix, MATLAB could not accurately take the inverse of Sigma0, which was crucial to the Bayesian update. This is flagged for future work in section 10.

Regardless, the Framework could still output weights:

A screen shot of a graph

Description automatically generated

Figure 24: output from BayesianPCA.mlx, BP2

This particular output was for BP2, and evaluated in Experiment 2.

The next section goes over experiment design:

## **4.1 Experiment Aims and Design:**

Considering the above mathematical background, many questions were posed as to the effectiveness of these mathematical techniques on the returns and risk of generated portfolios:

1. How would PCA subsampling on the Returns Matrix affect the output portfolio weights of the mean-variance optimization? Would it increase estimated/actual returns and decrease estimated/actual risk due to a reduction in noise?
2. How would Bayesian inferred posterior expected returns and risk estimators be different to standard Markowitz estimators? would it be beneficial from an investor looking to reduce risk and increase returns given limited financial data?
3. Can PCA subsampling and Bayesian Inference be combined to affect expected risk and return estimators and ensure that inputs into the mean-variance optimization framework ensure that the algorithm is more robust and provides a more accurate estimate of trends in returns and volatility of each asset? Post which, In a rolling investment scenario, can doing this increase the average monthly returns and decrease the risk?

Considering the above questions, two experiments were designed considering this data set:

## **5.1 The Data Set**

In order to evaluate the performance of the various Optimization frameworks and by extension, their estimators, the monthly closing price of ten stocks from the Dow-Jones Index were chosen. The closing prices were taken from 01/01/2010 to 01/12/2020.

This range was selected as it falls directly between two global crises: The 2008 US housing crisis and the 2020 Covid-19 Pandemic. In these periods, global economies widely fluctuated to due reduction in consumer buying power or consumer buying restrictions. Stock trends in these periods were heavily attuned by human response, inducing extremely volatile stock prices. These responses are inherently hard to quantify, and unexpected, and modern portfolio theory cannot account for such economic meltdowns. The market had recovered from the US housing Crisis in 2008 by the start of 2010, and the Market began to fall by March 2020 due to Covid, right after the project data set bounds. Thus, the data set was not affected by these crises, and trendlines can be inferred via proposed estimators explored below.

The following stocks were selected:

Table 1: Companies Selected

|  |  |
| --- | --- |
| **Index Name** | **Company Name** |
| LMT | Lockheed Martin Corporation |
| MSFT | Microsoft Inc. |
| APPL | Apple inc. |
| AMZN | Amazon inc. |
| V | Visa Inc. |
| MCD | McDonald’s Corporation |
| WFC | Wells Fargo and Co |
| CMCSA | Comcast Corp |
| MS | Morgan Stanley |
| C | Citigroup Inc |

These ten stocks were chosen as they branch from infrastructure (such as Amazon and Visa) to Consumables (McDonalds’s) to Technology (such as Microsoft and Apple.) Having a wide range of type of company allows for the optimisation analysis to be more holistic and not wholly affected by global trends at the time.

Figure 25: Stock price evolution of the 10 companies from 2010 to 2020

Above, figure 1 presents the evolution of stock prices from the 1st of January 2010 to the end of 2020. Largely, except for LMT, the stock prices of assets progress together. It was debated whether LMT should be retained in the data set as it showed severe variation, with the stock price ballooning from 140 dollars in the end of 2013 to 357 in March 2018. It was ultimately retained in order to explore how the optimization approaches would handle such a case.

This data was taken from Yahoo Finance Historical Data webpages [4] and collated into excel. This excel, fulldata.xlsx, can be found in the github. From there, the excel was inputted to each of the frameworks.

Visualizing the returns over time:

Figure 26: Monthly returns of Assets from 2010 to 2020

Ultimately, not much can be inferred visually from the frankly messy plot. However, via MATLAB, the returns were handled efficiently and mean returns and variance over various periods were assessed through mean-variance optimization logic.

The next section presents the experiment designs:

## **6.1 Experiment 1 Aims and Design:**

To emulate typical SIP funds, this experiment will focus on how the generated parameters are varied in a dynamic management scenario.

Consider this timeline:

A number and line with numbers

Description automatically generated with medium confidence

Figure 27: timeline for buypoints, Experiment 1

The figure above refers to a 3 year dynamic scenario where an investor invests in the beginning of 2015 based on initial portfolio weights given data, and utilizing the same capital under gain or loss, reshuffles the portfolio weights at BP2 and BP3 based on that points generated portfolio weights which are affected by variations in historical data. The table below details what historical data will be available to the algorithms. Please note due to the bug experienced in the Bayesian PCA framework, it was omitted from testing:

Table 2: Available data, exp1

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Buy Point** | **BP** | **Markowitz** | **PCA 95** | **Bayesian** |
| **2015** | **BP1** | 2010-2014 | 2010-2014 | 2010-2014 |
| **2016** | **BP2** | 2011-2015 | 2011-2015 | 2010-2014, 2015 |
| **2017** | **BP3** | 2012-2016 | 2012-2016 | 2011-2015, 2016 |

Here, the Buy Points Correspond to the year over which the portfolio weights are held, and the last 4 sections detail the beginning of the year to the beginning of the end year over which information is available. For instance, 2010-2014 would represent information available from 01/01/2010 till 01/01/2015.

The following portfolio derivations will be assessed across each buy period:

1. Derivation of portfolio weights via Mean-Variance Optimization with estimators for expected returns and risk being sampled from the returns of the assets over data available corresponding to each buy point. This follows Markowitz’s portfolio theory and serves as the baseline for the experiment.
2. Derivation of portfolio weights via Mean-Variance Optimization with estimators for expected returns and risk being sampled from the PCA denoised returns of the assets over information available at the corresponding buy point. This is similar to Markowitz’ portfolio theory but uses PCA at a threshold of 95% to denoise the returns matrix. It was evaluated whether a lower PCA threshold should be evaluated, but as the project author lacks in-depth knowledge of financial markets and their movements, it was decided a safer strategy would be employed potentially at the risk of negligible difference. The effect of lower PCA thresholds are detailed in Experiment 2.
3. Derivation of portfolio weights via Mean-Variance Optimization with estimators for expected returns and risk being sampled from the returns of the assets over information comprising of the prior buy periods ‘Markowitz’s information, representing the prior belief of estimators. This is then updated via Bayesian inference with the data of the year immediately preceding the buy point (observed data.) From this, a posterior estimator will be derived and used to infer portfolio weights for the corresponding buy period.

It will be assumed that the investor wishes to maximize the returns/risk ratio of the portfolio and has 1000$ to spread over assets as suggested by derived portfolio weights via the methods above. The evolution of the 1000$ will be tracked across each successive buy point, with the capital for the next buy point being the capital with excess returns from the evolution of returns during the previous buy period.

## **6.2 Experiment 2 Aims and Design:**

This experiment will consider an investor looking to maximize the sharpe ratio of his portfolio over the year 2016 (BP2). Available to him are the returns of assets from the beginning of 2010 to the end of 2015.

This experiments analysis is not concerned with the potential return/risk of portfolios generated in this time frame, but rather the behaviour of parameter estimators and their genesis via the four methods detailed below:

The following investment strategies will be evaluated:

1. Derivation of portfolio weights via Mean-Variance Optimization with estimators for expected returns and risk being sampled from the returns of the assets over 2011 to 2015. This follows Markowitz’s portfolio theory and serves as the baseline for the experiment.
2. Derivation of portfolio weights via Mean-Variance Optimization with estimators for expected returns and risk being sampled from the PCA denoised returns of the assets over 2011 to 2015. This is similar to Markowitz’ portfolio theory but evaluates the impact of PCA denoising over various cumulative variance (of Principal Components) thresholds on the expected returns and risk estimators. The cumulative variance thresholds investigated will be 95%, 90% and 85%.
3. Derivation of portfolio weights via Mean-Variance Optimization with estimators for expected returns and risk being sampled from the returns of the assets over 2010 to 2015, and then updated via Bayesian inference with 2016’s expected returns and risk. This will evaluate the impact of Bayesian inference on the portfolio weights, and the expected returns and risk.
4. Derivation of portfolio weights via Mean-Variance Optimization with estimators for expected returns and risk being sampled from the PCA denoised returns of the assets over 2010 to 2015, and then updated via Bayesian inference with 2016’s expected returns and risk. This will evaluate the impact of Bayesian inference on the portfolio weights, and the expected returns and risk. The threshold for the cumulative variance will be the best performing threshold from the 2nd Derivation.

It will be assumed that the investor wishes to maximize the returns/risk ratio of the portfolio and has 1000$ to spread over assets as suggested by derived portfolio weights via the methods above. The evolution of the 1000$ will be tracked across the monthly returns for 2016-2017.

## **7.2.1 Experiment 1 Results:**

This section considers an dynamic investment scenario. An investor looking to maximize his sharpe ratio for his portfolio has 1000 dollars to spread across the 10 assets. The initial buy point is at the beginning of 2015, the second buy point is at 2016, where portfolio weights will be reoptimized given new data, and accordingly, the third buy point is in 2017:

A number and line with numbers

Description automatically generated with medium confidence

Figure 28: BP timeline, EXP 1

Table 3: Available data, EXP1

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Buy Point** | **BP** | **Markowitz** | **PCA** | **Bayesian** |
| **2015** | **BP1** | 2010-2014 | 2010-2014 | 2010-2014 |
| **2016** | **BP2** | 2011-2015 | 2011-2015 | 2010-2014, 2015 |
| **2017** | **BP3** | 2012-2016 | 2012-2016 | 2011-2015, 2016 |

The above table highlights information available to the portfolio generators at each buy point. The data corresponding to each buy point was fed into the parameter estimators, and the following portfolio weights were found:

Markowitz:

Table 4: Markowitz Portfolios, EXP1

|  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Markowitz** | **BP** | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| **2015** | **BP1** | 0.2674 | 0.0000 | 0.1734 | 0.0184 | 0.1717 | 0.2331 | 0.0000 | 0.1359 | 0.0000 | 0.0000 |
| **2016** | **BP2** | 0.4014 | 0.0010 | 0.0368 | 0.1238 | 0.4366 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| **2017** | **BP3** | 0.5074 | 0.0753 | 0.0000 | 0.1336 | 0.1587 | 0.0000 | 0.0000 | 0.1245 | 0.0005 | 0.0000 |

PCA 95:

Table 5: PCA 95 portfolios, EXP1

|  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **PCA 95** | **BP** | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| **2015** | **BP1** | 0.2849 | 0.0000 | 0.1629 | 0.0219 | 0.1423 | 0.2394 | 0.0000 | 0.1486 | 0.0000 | 0.0000 |
| **2016** | **BP2** | 0.4011 | 0.0005 | 0.0381 | 0.1218 | 0.4384 | 0.0002 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| **2017** | **BP3** | 0.5041 | 0.0784 | 0.0000 | 0.1356 | 0.1521 | 0.0000 | 0.0001 | 0.1297 | 0.0000 | 0.0000 |

Bayesian:

Table 6: Bayesian Portfolios, EXP1

|  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Bayesian** | **BP** | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| **2015** | **BP1** | 0.2674 | 0.0000 | 0.1734 | 0.0184 | 0.1717 | 0.2331 | 0.0000 | 0.1359 | 0.0000 | 0.0000 |
| **2016** | **BP2** | 0.0000 | 0.0000 | 0.0000 | 1.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| **2017** | **BP3** | 0.3636 | 0.1786 | 0.0000 | 0.0008 | 0.1088 | 0.0000 | 0.0000 | 0.2789 | 0.0694 | 0.0000 |

The Markowitz and PCA 95 Portfolios very closely resembled each other, but the PCA 95 portfolio achieved more diversification as the highest weightage for PCA 95 was always lower than Markowitz except in the case of BP1 where PCA 95 assigned a weight of 0.2849 to LMT compared to 0.2674 for Markowitz. The Bayesian portfolio was widely different. Note that the Bayesian BP1 portfolio is the same as the Markowitz BP1 portfolio. This is because at the time, there was no new observed data for a posterior distribution to be inferred.

However, due to the widely different parameter estimation technique Bayesian inference used, in the BP2 period, expected returns for an asset often came up negative. This is because more importance was given to the most recent observed data, and in this case, 2016, the markets faced economic turmoil due to various external factors; the returns were much lower during these years than the ones preceding it. This can be seen in the excel main.xlsx file. As such, the mean-variance optimization program, limited by the no short selling rule, could only allocate weightage to those with positive expected posterior returns, resulting in widely undiversified portfolio weightages. BP2 is examined further in Experiment 2.

## **7.2.2 Portfolio Value Tracking:**

This section will evaluate for each buy period, the performance of each portfolio both in sample and the year the portfolio performs in out sample.

To begin with, BP1 corresponding to year 2015 is evaluated. The portfolio weights generated by each framework are as follows:

Table 7: BP1 Portfolios

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Framework** | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| **Markowitz** | 0.2674 | 0 | 0.1734 | 0.0184 | 0.1717 | 0.2331 | 0 | 0.1359 | 0 | 0 |
| **PCA 95** | 0.2849 | 0 | 0.1629 | 0.0219 | 0.1423 | 0.2394 | 0 | 0.1486 | 0 | 0 |
| **Bayesian** | 0.2674 | 0 | 0.1734 | 0.0184 | 0.1717 | 0.2331 | 0 | 0.1359 | 0 | 0 |

Note that as the Bayesian Framework at the time did not have any new information, there was no observed data to derive a distribution and subsequently, estimators for risk and return. Thus it has the same weights as the Markowitz Framework.

The in-sample data corresponds to the returns for the period 2010 to 2014. All testing for the portfolio weights were done in results.xlsx, also found in the GitHub. The in-sample performance is as follows. For all in sample testing, now and henceforth, 1000 Dollars is taken as the start point. For in-sample data corresponding to BP1, the following was observed:

Table 8: Insample perfomance, BP1

|  |  |  |  |
| --- | --- | --- | --- |
| **Metric** | **Markowitz** | **PCA 95** | **Bayesian** |
| **Investment** | 999.9 | 1000 | 999.9 |
| **Final Value** | 2830.812984 | 2804.476718 | 2830.812984 |
| **Profit** | 1830.912984 | 1804.476718 | 1830.912984 |
| **ROI** | 1.831096094 | 1.804476718 | 1.831096094 |

Please note that all decimal lengths were left as is, as comparisons throughout the project become very minute.

For the in-sample data, The Markowitz framework provided a 1.831 return on investment compared to PCA 95’s 1.804. The same was observed for the Bayesian portfolio given it was the same as the Markowitz portfolio. Comparing the average monthly returns and risk:

Table 9: Insample metrics, BP1

|  |  |  |  |
| --- | --- | --- | --- |
| **Metric** | **Markowitz** | **PCA 95** | **Bayesian** |
| **expected return** | 0.01837 | 0.018194 | 0.01837 |
| **risk** | 0.034405 | 0.033992 | 0.034405 |
| **variance** | 0.001184 | 0.001155 | 0.001184 |
| **sharpe ratio** | 0.533948 | 0.535237 | 0.533948 |

From here, the Markowitz (and Bayesian) portfolio yielded the highest return and sharpe ratio. Recall that the report considers results from the perspective of a risk adjusted return investor looking to maximize the sharpe ratio. This indicates that the Markowitz portfolio may be more overfitted to the in-sample data (and by extension, estimators) than PCA 95, which was the intended effect. Visualizing the performance of the portfolio:

Figure 29: Visualized BP1 insample performance

The difference between the Markowitz (and Bayesian) and PCA 95 portfolio evolution in sample is virtually indistinguishable. This suggests that a lower threshold for PCA may have been more apt for evaluation. Still an investor wouldn’t be as concerned about the in-sample performance of a portfolio as hindsight is always 20/20. Comparing this methods out-of-sample is key, and as such, the out-sample performance is as follows:

Table 10: Outsample performance, BP1

|  |  |  |  |
| --- | --- | --- | --- |
| **Date** | **Markowitz** | **Bayesian** | **PCA 95** |
| **31/12/2014** | 1000 | 1000 | 1000 |
| **01/01/2015** | 988.0444 | 988.0444 | 987.2847 |
| **01/02/2015** | 1064.939 | 1064.939 | 1064.415 |
| **01/03/2015** | 1045.39 | 1045.39 | 1045.771 |
| **01/04/2015** | 1029.233 | 1029.233 | 1028.555 |
| **01/05/2015** | 1046.942 | 1046.942 | 1044.937 |
| **01/06/2015** | 1034.206 | 1034.206 | 1033.505 |
| **01/07/2015** | 1101.894 | 1101.894 | 1101.809 |
| **01/08/2015** | 1042.012 | 1042.012 | 1042.094 |
| **01/09/2015** | 1052.325 | 1052.325 | 1054.279 |
| **01/10/2015** | 1159.246 | 1159.246 | 1161.376 |
| **01/11/2015** | 1163.111 | 1163.111 | 1164.834 |
| **01/12/2015** | 1136.191 | 1136.191 | 1139.05 |
| **Metric** | **Markowitz** | **Bayesian** | **PCA 95** |
| **Investment** | 1000 | 1000 | 1000 |
| **Final Value** | 1136.191 | 1136.191 | 1139.05 |
| **Profit** | 136.1914 | 136.1914 | 139.0498 |
| **ROI** | 0.136191 | 0.136191 | 0.13905 |

With an initial investment of 1000 dollars in the beginning of 2015, an investor following weights suggested by the Markowitz portfolio would walk away with 136.19 dollars of profit whilst the PCA 95 portfolio put forth 139.05 dollars resulting in an ROI of 13.905% compared to Markowitz’s 13.61%. Examining the monthly returns:

Table 11: Out sample metrics, BP1

|  |  |  |  |
| --- | --- | --- | --- |
| **Date** | **Markowitz** | **Bayesian** | **PCA 95** |
| **01/01/2015** | -0.011956 | -0.011956 | -0.012715 |
| **01/02/2015** | 0.077825 | 0.077825 | 0.078123 |
| **01/03/2015** | -0.018357 | -0.018357 | -0.017515 |
| **01/04/2015** | -0.015455 | -0.015455 | -0.016462 |
| **01/05/2015** | 0.017206 | 0.017206 | 0.015927 |
| **01/06/2015** | -0.012165 | -0.012165 | -0.010941 |
| **01/07/2015** | 0.065449 | 0.065449 | 0.066090 |
| **01/08/2015** | -0.054344 | -0.054344 | -0.054197 |
| **01/09/2015** | 0.009897 | 0.009897 | 0.011692 |
| **01/10/2015** | 0.101605 | 0.101605 | 0.101583 |
| **01/11/2015** | 0.003334 | 0.003334 | 0.002978 |
| **01/12/2015** | -0.023145 | -0.023145 | -0.022136 |
| **Metric** | **Markowitz** | **Bayeisan** | **PCA 95** |
| **expected ret** | 0.011657843 | 0.011658 | 0.011869 |
| **risk** | 0.046508963 | 0.046509 | 0.065516 |
| **variance** | 0.002163084 | 0.002163 | 0.004292 |
| **sharpe ratio** | 0.250657987 | 0.250658 | 0.181161 |

Here, despite having a higher realised return than Markowitz at 0.011869 compared to 0.011657 for PCA 95, the risk (standard deviation) was higher at 0.065516 compared to 0.0465000. This is important because the sharpe ratio was ultimately impacted, yielding 0.81161 compared to 0.2506567 for PCA95 and Markowitz respectively. While it is true PCA95 had higher returns, consider an investor actively looking at the performance of his portfolio. At points such as 01/06/2015, the return was 0.002 lower than Markowitz. The sharpe ratio represents the return by risk, and though a higher risk means that at any point the return may exceed the average return, it also means that the return could fall the same amount. The average return indicates however, the average monthly increase of that stock over time, in this case, the year 2015. Thus, the sharpe ratio represents the balance between return and risk, and a risk adjusted investor would always seek to increase this.

Evaluating the portfolio progression visually:

Figure 30: BP1 out sample performance visualized

There is no immediate perceptible difference, except for the latter section from 09/2015 onwards, where the Markowitz/Bayesian line falls under PCA 95.

## **7.2.3 Portfolio Value Tracking: BP2**

Buy period 2, corresponding to the year 2016, is interesting as now, the Bayesian framework has observed data to formulate posterior estimators. The following data was used to estimate the weights:

Table 12: information avaiable, BP2

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Buy Point** | **BP** | **Markowitz** | **PCA 95** | **Bayesian**  **(prior, observed)** |
| **2016** | **BP2** | 2011-2015 | 2011-2015 | 2010-2014, 2015 |

Via the frameworks, the following weights were retrieved:

Table 13: Portfolio weights, BP2

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Framework** | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| **Markowitz** | 0.4014 | 0.001 | 0.0368 | 0.1238 | 0.4366 | 0.0003 | 0 | 0 | 0 | 0 |
| **PCA 95** | 0.4011 | 0.0005 | 0.0381 | 0.1218 | 0.4384 | 0.0002 | 0 | 0 | 0 | 0 |
| **Bayesian** | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |

There was minor variances between Markowitz and PCA 95, but notably, the weightage increased for V and decreased for LMT between PCA 95 and Markowitz. The Bayesian portfolio was rather stunning, dedicating its entire weight to AMZN. An analysis of why this happened is in Experiment 2. For now, these weights are carried forward.

The in-sample data corresponds to the returns for the period 2011 to 2015. The in-sample performance is as follows. For in-sample data corresponding to BP2, the following was observed:

Table 14: in sample performance, BP2

|  |  |  |  |
| --- | --- | --- | --- |
| **Metric** | **Markowitz** | **PCA 95** | **Bayesian** |
| **Investment** | 1000 | 1000 | 1000 |
| **Final Value** | 3722.49213 | 3723.805663 | 3754.944222 |
| **Profit** | 2722.49213 | 2723.805663 | 2754.944222 |
| **ROI** | 2.72249213 | 2.723805663 | 2.754944222 |

For the in-sample data, The Markowitz framework provided a 2.722 return on investment compared to PCA 95’s 2.724. The Bayesian method provided a higher return at 2.755. Comparing the average monthly returns and risk:

Table 15: insample metrics, BP2

|  |  |  |  |
| --- | --- | --- | --- |
| **Metric** | **Markowitz** | **PCA 95** | **Bayesian** |
| **expected ret** | 0.021618 | 0.021626 | 0.025319 |
| **risk** | 0.035567 | 0.035572 | 0.080659 |
| **variance** | 0.001265 | 0.001265 | 0.006506 |
| **sharpe ratio** | 0.60779 | 0.607947 | 0.313902 |

The in-sample monthly returns for the Bayesian method were significantly higher than both Markowitz and PCA 95 at 0.025319 compared to 0.021618 and 0.021626, however the risk experienced was also much higher at 0.080659 compared to 0.035567/0.035572. As a result, the sharpe ratio of the Bayesian portfolio was nearly half of Markowitz/PCA 95 at 0.313902. As such, this portfolio strategy may be more inline for return oriented investors, who would not mind the additional risk at the prospect of higher returns.

Visualizing the in-sample performance:

Figure 31: In-sample performance, visualized

Though the Bayesian line ended up above Markowitz and PCA 95, it was not without its caveats, for the majority of time, it remained below the Markowitz and PCA 95 lines and experienced major volatility especially at the peak in 12/2013.

Still, an investor would be concerned in how the portfolio weights translate out of sample. Using the final investments for each portfolio from BP1, the portfolio valuation evolved as follows:

Table 16: Outsample performance, BP2

|  |  |  |  |
| --- | --- | --- | --- |
| **Date** | **Markowitz** | **Bayesian** | **PCA 95** |
| **31/12/2015** | 1136.078 | 1136.191 | 1139.164 |
| **01/01/2016** | 1081.951 | 986.7647 | 1085.016 |
| **01/02/2016** | 1071.039 | 928.8028 | 1074.17 |
| **01/03/2016** | 1122.751 | 997.9267 | 1126.093 |
| **01/04/2016** | 1158.022 | 1108.791 | 1161.074 |
| **01/05/2016** | 1192.618 | 1215.032 | 1195.64 |
| **01/06/2016** | 1183.956 | 1202.979 | 1186.796 |
| **01/07/2016** | 1230.865 | 1275.582 | 1233.841 |
| **01/08/2016** | 1231.512 | 1292.981 | 1234.566 |
| **01/09/2016** | 1253.146 | 1407.543 | 1256.174 |
| **01/10/2016** | 1256.269 | 1327.711 | 1259.435 |
| **01/11/2016** | 1253.36 | 1261.731 | 1256.432 |
| **01/12/2016** | 1227.689 | 1260.554 | 1230.793 |
| **Metric** | **Markowitz** | **Bayesian** | **PCA 95** |
| **Investment** | 1136.078 | 1136.191 | 1139.164 |
| **Final Value** | 1227.689 | 1260.554 | 1230.793 |
| **Profit** | 91.61081 | 124.3627 | 91.62949 |
| **ROI** | 0.080638 | 0.109456 | 0.080436 |

Please note that Excel introduced slight errors due to decimal error. As such, the difference between Markowitz and Bayesian initial investments had a slight difference of 0.113 when the should’ve been the same. Nevertheless, the in-sample performance in this case tracked over to the out-sample, resulting in the Bayesian portfolio having the highest ROI of 0.109 compared to Markowitz and PCA 95 of 0.080638 and 0.080436. Examing the monthly returns trends:

Table 17: Outsample Metrics, BP2

|  |  |  |  |
| --- | --- | --- | --- |
| **Date** | **Markowitz** | **Bayesian** | **PCA 95** |
| **01/01/2016** | -0.047644 | -0.131515 | -0.047533 |
| **01/02/2016** | -0.010085 | -0.058739 | -0.009996 |
| **01/03/2016** | 0.048282 | 0.074423 | 0.048337 |
| **01/04/2016** | 0.031416 | 0.111094 | 0.031064 |
| **01/05/2016** | 0.029874 | 0.095817 | 0.029771 |
| **01/06/2016** | -0.007263 | -0.009920 | -0.007397 |
| **01/07/2016** | 0.039621 | 0.060353 | 0.039640 |
| **01/08/2016** | 0.000526 | 0.013640 | 0.000588 |
| **01/09/2016** | 0.017567 | 0.088603 | 0.017502 |
| **01/10/2016** | 0.002492 | -0.056717 | 0.002596 |
| **01/11/2016** | -0.002315 | -0.049695 | -0.002385 |
| **01/12/2016** | -0.020482 | -0.000933 | -0.020406 |
| **Metric** | **Markowitz** | **Bayeisan** | **PCA 95** |
| **expected ret** | 0.006832304 | 0.011368 | 0.006815 |
| **risk** | 0.027608629 | 0.076111 | 0.027554 |
| **variance** | 0.000762236 | 0.005793 | 0.000759 |
| **sharpe ratio** | 0.247469888 | 0.149354 | 0.247343 |

Here, the margin between the sharpe ratio for PCA 95 and Markowitz was small, yet curiously, despite having a higher sharpe ratio in-sample, PCA 95 did not perform as well as Markowitz, resulting in marginally lower monthly returns. However, the risk experienced was still lower than Markowitz, as in the case of BP1. The Bayesian portfolio presents a very different story. Though the monthly returns were much higher at 0.011368, almost double that of Markowitz and PCA 95, it came at a much higher risk, nearly triple that of Markowitz and PCA 95 at 0.07611. This has impacted the sharpe performance ratio severely standing at 0.149354. However, recall that the ROI was much higher than Markowitz and PCA95, and so, the Bayesian portfolio strategy is seemingly more inclined towards more return-oriented investors. Visualizing the performance through BP2:

Figure 32: Out sample performance Visualized, BP2

Once again, the visual difference between the Markowitz and PCA 95 line are negligible. This suggests that the threshold of 95% for cumulative variance might have been too low for proper evaluation. There are several aspects to that, and this is detailed in the evaluations and conclusions section. The Bayesian line however, was far more volatile that the PCA 95/Markowitz line, represented by it being under at 01/2016 and then above in 2016. This was a very curious case, considering the performance and weightage allocation, and was chosen for Experiment 2 to examine the behaviour of each portfolio generation framework.

## **7.2.5 Portfolio Value Tracking: BP3**

Now, BP3 corresponding to year 2017 is evaluated. The following information was available to each framework:

Table 18: information available, BP3

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Buy Point** | **BP** | **Markowitz** | **PCA 95** | **Bayesian**  **(prior, observed)** |
| **2017** | **BP3** | 2012-2016 | 2012-2016 | 2011-2015, 2016 |

The portfolio weights generated by each framework are as follows:

Table 19: Portfolios, BP3

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Framework** | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| **Markowitz** | 0.5074 | 0.0753 | 0.0000 | 0.1336 | 0.1587 | 0.0000 | 0.0000 | 0.1245 | 0.0005 | 0.0000 |
| **PCA 95** | 0.5041 | 0.0784 | 0.0000 | 0.1356 | 0.1521 | 0.0000 | 0.0001 | 0.1297 | 0.0000 | 0.0000 |
| **Bayesian** | 0.3636 | 0.1786 | 0.0000 | 0.0008 | 0.1088 | 0.0000 | 0.0000 | 0.2789 | 0.0694 | 0.0000 |

Once again, Markowitz and PCA 95 had similar weights, with PCA 95 having a more diverse portfolio corresponding to the reduction in weight of LMT from 0.5074 to 0.5041. The Bayesian method went even further, staking more weight on MS and MSFT, and reducing LMT’s weight to 0.3636.

The in-sample data corresponds to the returns for the period 2012 to 2016. The in sample performance is as follows. As in prior cases, 1000 dollars was taken as the start point:

Table 20: In sample performance, BP3

|  |  |  |  |
| --- | --- | --- | --- |
| **Metric** | Markowitz | **PCA 95** | **Bayesian** |
| **Investment** | 1000 | 1000 | 1000 |
| **Final Value** | 3178.390711 | 3177.940472 | 2894.7735 |
| **Profit** | 2178.390711 | 2177.940472 | 1894.7735 |
| **ROI** | 2.178390711 | 2.177940472 | 1.8947735 |

For the BP3 in-sample data, The Markowitz framework provided a 2.178 return on investment. PCA performed similarly. The Bayesian portfolio however, struggled to compare to Markowitz and PCA 95, notching only 1.894 ROI. Comparing the in sample returns and risk:

Table 21: In sample metrics, BP3

|  |  |  |  |
| --- | --- | --- | --- |
| **Metric** | **Markowitz** | **PCA 95** | **Bayesian** |
| **expected ret** | 0.019292 | 0.019287 | 0.017256 |
| **risk** | 0.03392 | 0.033929 | 0.034624 |
| **variance** | 0.001151 | 0.001151 | 0.001199 |
| **sharpe ratio** | 0.568755 | 0.568456 | 0.498379 |

From above, the Bayesian portfolio performed the worst, with the highest risk and lowest insample monthly returns. The sharpe ratio accordingly was worse at 0.498379 compared to Markowitz and PCA 95 at 0.568456/0.568755. This indicates that Markowitz and PCA 95 were fit better to the in-sample returns, which may not necessarily indicate better performance out sample.

Visualizing the in-sample performance:

Figure 33: BP3 insample perfomance visualized

Here, the Markowitz line and PCA 95 line had virtually no difference. The Bayesian line also closely resembled the but begins to deviate in the beginning of 2015. This is potentially due to the presence of 0.2789 of CMCSA, which did not perform well in 2015. But, the Bayesian framework assigned weight to it, and examining the out sample performance is more important: Using the carried forward final prices of BP2, the frameworks achieved the following portfolio valuation projection:

Table 22: Out sample performance, BP3

|  |  |  |  |
| --- | --- | --- | --- |
| **Date** | **Markowitz** | **Bayesian** | **PCA 95** |
| **31/12/2016** | 1227.689 | 1260.554 | 1230.793 |
| **01/01/2017** | 1276.804 | 1313.596 | 1280.507 |
| **01/02/2017** | 1330.353 | 1351.988 | 1333.333 |
| **01/03/2017** | 1347.804 | 1358.232 | 1351.033 |
| **01/04/2017** | 1377.652 | 1392.617 | 1381.253 |
| **01/05/2017** | 1445.104 | 1448.261 | 1449.062 |
| **01/06/2017** | 1413.311 | 1413.556 | 1416.709 |
| **01/07/2017** | 1480.676 | 1484.454 | 1483.95 |
| **01/08/2017** | 1525.81 | 1521.951 | 1528.717 |
| **01/09/2017** | 1528.3 | 1516.759 | 1530.449 |
| **01/10/2017** | 1568.418 | 1531.704 | 1570.738 |
| **01/11/2017** | 1626.065 | 1578.783 | 1628.634 |
| **01/12/2017** | 1646.017 | 1617.13 | 1648.978 |
| **Metric** | **Markowitz** | **PCA 95** | **Bayesian** |
| **Investment** | 1227.689 | 1260.554 | 1230.793 |
| **Final Value** | 1646.017 | 1617.13 | 1648.978 |
| **Profit** | 418.3281 | 356.5755 | 418.1847 |
| **ROI** | 0.340744 | 0.282872 | 0.339768 |

Here, Markowitz and PCA 95 once again performed very similarly, with Markowitz offering a higher return on investment at 0.3407 compared to PCA 95s 0.2828. The Bayesian portfolio also performed well, with a return of 0.3398.

Examining the monthly returns:

Table 23: Out sample metrics, BP3

|  |  |  |  |
| --- | --- | --- | --- |
| **Date** | **Markowitz** | **Bayesian** | **PCA 95** |
| **01/01/2017** | 0.040006 | 0.042078 | 0.040392 |
| **01/02/2017** | 0.041940 | 0.029227 | 0.041254 |
| **01/03/2017** | 0.013117 | 0.004618 | 0.013275 |
| **01/04/2017** | 0.022146 | 0.025316 | 0.022368 |
| **01/05/2017** | 0.048961 | 0.039956 | 0.049093 |
| **01/06/2017** | -0.022000 | -0.023963 | -0.022327 |
| **01/07/2017** | 0.047665 | 0.050156 | 0.047462 |
| **01/08/2017** | 0.030482 | 0.025260 | 0.030168 |
| **01/09/2017** | 0.001632 | -0.003412 | 0.001133 |
| **01/10/2017** | 0.026250 | 0.009853 | 0.026324 |
| **01/11/2017** | 0.036755 | 0.030737 | 0.036859 |
| **01/12/2017** | 0.012270 | 0.024289 | 0.012492 |
| **Metric** | **Markowitz** | **PCA 95** | **Bayesian** |
| **expected ret** | 0.024935329 | 0.021176 | 0.024874 |
| **risk** | 0.02096803 | 0.021049 | 0.021031 |
| **variance** | 0.000439658 | 0.000443 | 0.000442 |
| **sharpe ratio** | 1.189207034 | 1.006053 | 1.182749 |

Here Markowitz had the highest Sharpe ratio performance as a result of having the lowest risk and highest realised monthly returns.

Evaluating the portfolio progression visually:

Figure 34: Out sample performance, BP3 Visualized

Once again, the PCA 95 line and the Markowitz line were virtually indistinguishable. The Bayesian line due to higher experienced volatility, experienced fluctuations resulting in performing better from 01/2017 to 04/2017, the dropping under Markowitz post 08/2017.

The next section will cover the entire valuation tracking through BP1, BP2, BP3

## **7.2.6 Portfolio Value Tracking: Overall, BP1, BP2, BP3**

Considering all BP stages, the portfolio with different weights suggested by each framework evolved like so:

Figure 35: Overall Portfolio Progression, BP1, BP2, BP3

The overall metrics were as follows:

Table 24: Overall metrics, Exp 1

|  |  |  |  |
| --- | --- | --- | --- |
|  | Markowitz | Bayesian | PCA 95 |
| Start, 2015 | 1000 | 1000 | 1000 |
| BP1 End | 1136.191 | 1136.191 | 1139.05 |
| BP2 End | 1227.689 | 1260.554 | 1230.793 |
| BP3 End | 1571.895 | 1350.481 | 1568.162 |
| Mean returns | 0.010399 | 0.007617 | 0.010332 |
| Variance | 0.001894 | 0.002715 | 0.00186 |
| Risk | 0.04352 | 0.052102 | 0.043133 |
| Sharpe Performace | 0.23896 | 0.146199 | 0.239544 |
| Total ROI | 1.571895 | 1.350481 | 1.568162 |
| Total Profit | 571.8947 | 350.4809 | 568.1615 |

Here, the Bayesian portfolio had the lowest sharpe performance. Again, the sharpe performance is important because it speaks to the performance of the portfolio at any point in time across the timeframe 2015-2018. Recall however, it had periods such as BP2 where it made big bets and had the most monthly returns. However, the big betting nature also inhibited its performance, where in BP3, it tanked heavily. This high risk high reward nature displayed by the Bayesian framework may be of use to more return oriented investors.

The Markowitz model performed well, notching a sharpe performance of 0.23896 and the highest average monthly returns at 0.10399. It also had the highest Return on Investment, yielding 571.89 dollars. However, the PCA 95 model more accurately predicted risky investments, resulting in a lower volatility of 0.00186 compared to Markowitz’s 0.001894. It also resulted in more diverse portfolios, as evidenced in section 7.1. Thus, as a robust strategy, the PCA 95 model performed the best, further evidenced by its Sharpe performance of 0.239544, the highest of the three frameworks.

The next section goes over the behaviour of all three frameworks and the faulty BayesianPCA1.mlx framework for BP2.

## **8.1.1 Experiment 2 Results:**

This section will go over portfolio weight selection and its subsequent performance for the buy period 2016-2017.

To begin with, Markowitz Mean Variance Optimization is considered:

## **8.1.2 Markowitz Optimized weights for buy period ‘16-‘17**

The table below presents the expected returns estimators and cumulative risk for each asset from the associated covariance matrix. This table is the average returns and risk for 2011-2016.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Parameter** | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| mean | 0.019963 | 0.013409 | 0.016312 | 0.025314 | 0.026128 | 0.007869 | 0.010189 | 0.017338 | 0.007566 | 0.005232 |
| Cum Var | 0.009293 | 0.017052 | 0.014637 | 0.016514 | 0.011076 | 0.006207 | 0.012561 | 0.016765 | 0.032962 | 0.029825 |
| Cum Risk | 0.096402 | 0.130583 | 0.120984 | 0.128507 | 0.105241 | 0.078782 | 0.112078 | 0.12948 | 0.181555 | 0.1727 |
| return/risk | 0.207084 | 0.102682 | 0.134828 | 0.196988 | 0.24827 | 0.09988 | 0.090913 | 0.133902 | 0.041675 | 0.030295 |

Recall that the Cumulative variance (Cum Var) is the sum of all covariance and variance for that asset. The square root of this represents the Cumulative Risk.

From this, the Mean-Variance framework detailed in Section 5 returned the following portfolio weights:

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| Markowitz | 0.4014 | 0.001 | 0.0368 | 0.1238 | 0.4366 | 0.0003 | 0 | 0 | 0 | 0 |

This portfolio weight spread came with an expected return of 0.231 and expected risk 0.0367 yielding an expected sharpe ratio of 0.6312.

These weights suggest that splitting mainly between LMT and V would be the most beneficial, with the both of them comprising 0.8380 of the budget weight. This tracks with the return/risk ratio above in table x, with both of them having a ratio of 0.208 and 0.248 respectively.

The performance of the portfolio weights are as follows:

|  |  |
| --- | --- |
| **Price** | **Markowitz** |
| 31/12/2015 | 1000 |
| 01/01/2016 | 952.2609792 |
| 01/02/2016 | 942.6569987 |
| 01/03/2016 | 988.1702331 |
| 01/04/2016 | 1019.214223 |
| 01/05/2016 | 1049.66265 |
| 01/06/2016 | 1042.038925 |
| 01/07/2016 | 1083.325713 |
| 01/08/2016 | 1083.89524 |
| 01/09/2016 | 1102.935778 |
| 01/10/2016 | 1105.684577 |
| 01/11/2016 | 1103.124385 |
| 01/12/2016 | 1080.529731 |

Thus, with an initial investment of 1000 dollars at the beginning of 2016, the investor would be left with 1080.52 dollars, a return of 8.052%. The average monthly returns (below) were found to be 0.0068 with a risk (represented as standard deviation) of 0.0263, yielding a sharpe performance ratio of 0.246, which was very different to the expected 0.6312. This is now considered the baseline for further results in the experiment.

|  |  |
| --- | --- |
| **Returns** | **Markowitz** |
| 01/01/2016 | -0.04774 |
| 01/02/2016 | -0.01009 |
| 01/03/2016 | 0.048282 |
| 01/04/2016 | 0.031416 |
| 01/05/2016 | 0.029874 |
| 01/06/2016 | -0.00726 |
| 01/07/2016 | 0.039621 |
| 01/08/2016 | 0.000526 |
| 01/09/2016 | 0.017567 |
| 01/10/2016 | 0.002492 |
| 01/11/2016 | -0.00232 |
| 01/12/2016 | -0.02048 |
| mean | 0.006824 |
| var | 0.000763 |
| risk | 0.027626 |

## **8.1.3 PCA reduced weights for buy period ‘16-‘17**

This section examines the impact of PCA at threshold variance levels of 95%, 90% and 85%. To begin with, the effect of PCA is examined on the returns of a single asset: AMZN. A single asset was considered as much like figure 2 in Section 4.1, the entire plot of all asset returns would be far too visually complex.

Figure 36: AMZN returns 11-16 no PCA

The graph presents the returns for Amazon over the beginning of 2011 to the beginning of 2016. Not much can be inferred but on a visual basis, no observable trend can be distinguished.

Figure 37: AMZN returns 11-16 PCA 95

At a threshold level of 95%, PCA begins to have some effect. The values of returns are not considered here as the differences were found to be minute, but the first two peak between 01/2011 and 10/2011 show some variance, the first peak is increased and the second is decreased, suggesting that there were both anomalous positive returns during the first year: 2011. Also to be considered is that PCA was done on the returns matrix as a whole, and AMZN is only one of ten assets. These plots are just mainly visual references to see the effect PCA on the returns matrix has on a singular asset.

Figure 38: AMZN returns 11-16 PCA 90

At PCA 90, there was no visual correspondence of any return changes across 01/2011 to 01/2016.

Figure 39: AMZN returns 11-16 PCA 85

At PCA 85, there were some changes seen from PCA 90. Most notably, the peaks at 01/2015 to 05/2015 were attenuated and brought in line with each other, implying the return at 01/2015 was anomalous.

Evaluating the change in data is another interesting aspect of the effectiveness of PCA.

Table 25: PCA 95

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Parameter** | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| mean | 0.019963 | 0.013409 | 0.016312 | 0.025314 | 0.026128 | 0.007869 | 0.010189 | 0.017338 | 0.007566 | 0.005232 |
| cum var | 0.00921 | 0.017085 | 0.014449 | 0.016661 | 0.011062 | 0.006105 | 0.012879 | 0.016895 | 0.032637 | 0.029499 |
| cum risk | 0.095968 | 0.130708 | 0.120203 | 0.129077 | 0.105174 | 0.078133 | 0.113484 | 0.12998 | 0.180658 | 0.171753 |
| return/risk | 0.20802 | 0.102583 | 0.135704 | 0.196118 | 0.248427 | 0.100709 | 0.089786 | 0.133387 | 0.041882 | 0.030462 |

Comparing this with the earlier Markowitz estimators:

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Parameter** | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| mean | 0.019963 | 0.013409 | 0.016312 | 0.025314 | 0.026128 | 0.007869 | 0.010189 | 0.017338 | 0.007566 | 0.005232 |
| Cum Var | 0.009293 | 0.017052 | 0.014637 | 0.016514 | 0.011076 | 0.006207 | 0.012561 | 0.016765 | 0.032962 | 0.029825 |
| Cum Risk | 0.096402 | 0.130583 | 0.120984 | 0.128507 | 0.105241 | 0.078782 | 0.112078 | 0.12948 | 0.181555 | 0.1727 |
| return/risk | 0.207084 | 0.102682 | 0.134828 | 0.196988 | 0.24827 | 0.09988 | 0.090913 | 0.133902 | 0.041675 | 0.030295 |

Not much change is inferred, however, the cumulative variances seem to have been affected, decreasing across the board except in some cases, namely APPL. This implies that by removing anomalous readings, the average returns were more accurately gauged, resulting in a lower volatility.

Below are the estimator tables for PCA 90 and 85:

Table 26: PCA 90

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Parameter** | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| mean | 0.019963 | 0.013409 | 0.016312 | 0.025314 | 0.026128 | 0.007869 | 0.010189 | 0.017338 | 0.007566 | 0.005232 |
| cum var | 0.009238 | 0.017125 | 0.014429 | 0.016671 | 0.011018 | 0.006103 | 0.012849 | 0.016968 | 0.032584 | 0.029484 |
| cum risk | 0.096113 | 0.130864 | 0.120119 | 0.129115 | 0.104966 | 0.078122 | 0.113354 | 0.130263 | 0.18051 | 0.17171 |
| return/risk | 0.207706 | 0.102462 | 0.135799 | 0.19606 | 0.248919 | 0.100723 | 0.089889 | 0.133097 | 0.041916 | 0.03047 |

Table 27: PCA 85

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Parameter** | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| mean | 0.019963 | 0.013409 | 0.016312 | 0.025314 | 0.026128 | 0.007869 | 0.010189 | 0.017338 | 0.007566 | 0.005232 |
| cum var | 0.009237 | 0.017142 | 0.014425 | 0.016667 | 0.011029 | 0.006094 | 0.012845 | 0.016966 | 0.032578 | 0.029486 |
| cum risk | 0.096108 | 0.130926 | 0.120104 | 0.1291 | 0.10502 | 0.078066 | 0.113336 | 0.130253 | 0.180493 | 0.171716 |
| return/risk | 0.207718 | 0.102413 | 0.135816 | 0.196083 | 0.248792 | 0.100796 | 0.089903 | 0.133107 | 0.04192 | 0.030468 |

And correspondingly, the portfolio weights generated for PCA 95, 90, 85:

Table 28: PCA portfolio weights 16-17

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| Markowitz | 0.4014 | 0.001 | 0.0368 | 0.1238 | 0.4366 | 0.0003 | 0 | 0 | 0 | 0 |
| PCA 95 | 0.4011 | 0.0005 | 0.0381 | 0.1218 | 0.4384 | 0.0002 | 0 | 0 | 0 | 0 |
| PCA 90 | 0.3665 | 0 | 0.0397 | 0.1167 | 0.4771 | 0 | 0 | 0 | 0 | 0 |
| PCA 85 | 0.2992 | 0.0439 | 0.0283 | 0.0772 | 0.5513 | 0 | 0 | 0 | 0 | 0 |

As seen above, as the PCA threshold increased, weights for LMT decreased from 0.4014 down to 0.2992 (Markowitz weights are essentially PCA 100.) This is interesting as though the expected return didn’t change, the cumulative risk did, going from 0.09597 to 0.096109. This corresponded to a drop in return risk ratio from 0.20802 to 0.207718. Conversely, V had an increasing risk/return ratio corresponding to a decrease in cumulative risk going from 0.105241 to 0.105200. this resulted in a increase of the risk/return ratio from 0.24827 to 0.248792, resulting in an increase of portfolio weight from 0.4366 to 0.5513.

Below are the tracked investment for Markowitz, PCA 95, PCA 90, PCA 85:

Table 29: returns Markowitz, PCA, 16-17

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Price | Marko | PCA 95 | PCA 90 | PCA 85 |
| 31/12/2015 | 1000 | **1000.1** | 1000 | 999.9 |
| **01/01/2016** | 952.2609792 | **952.562407** | 952.4595925 | 957.0818289 |
| **01/02/2016** | 942.6569987 | 943.0405868 | 941.4259716 | 941.3085782 |
| **01/03/2016** | 988.1702331 | 988.6247082 | 987.9644388 | 989.6975094 |
| **01/04/2016** | 1019.214223 | 1019.335434 | 1016.638336 | 1009.282764 |
| **01/05/2016** | 1049.66265 | 1049.682114 | 1046.815364 | 1038.013533 |
| **01/06/2016** | 1042.038925 | 1041.91773 | 1034.776861 | 1017.14295 |
| **01/07/2016** | 1083.325713 | 1083.21943 | 1077.048113 | 1062.6167 |
| **01/08/2016** | 1083.89524 | 1083.855963 | 1080.608987 | 1071.736946 |
| **01/09/2016** | 1102.935778 | 1102.825938 | 1100.585719 | 1089.818453 |
| **01/10/2016** | 1105.684577 | 1105.689155 | 1102.642543 | 1094.187472 |
| **01/11/2016** | 1103.124385 | 1103.05254 | 1094.645113 | 1078.265911 |
| **01/12/2016** | 1080.529731 | 1080.543796 | 1074.989891 | 1064.979886 |

Interestingly, PCA 95 performed only extremely marginally better than Markowitz, whilst PCA 90 and PCA 95 both performed worse with returns of 1074.99 and 1064.98 respectively. It should be noted there were some slight rounding errors, namely in PCA.95 and PCA 85 both of which went over and under the 1000$ budget by 10 cents. This immediately discredits PCA 95 performing better than PCA as the net gain was only 2 cents over Markowitz, however

The expected returns, risk and sharpe ratio taken from the mean-variance framework were as follows:

|  |  |  |  |
| --- | --- | --- | --- |
|  | **Expected Return** | **expected risk** | **expected sharpe** |
| Marko | 0.023173 | 0.036706 | 0.631313682 |
| PCA 95 | 0.023173 | 0.036673 | 0.631881766 |
| PCA 90 | 0.023384 | 0.036916 | 0.633438076 |
| PCA 85 | 0.023384 | 0.036106 | 0.64764859 |

And the subsequent monthly return performance was as follows:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Returns | Marko | PCA 95 | PCA 90 | PCA 85 |
| **01/01/2016** | -0.047739021 | -0.04753284 | -0.04754 | -0.04282 |
| **01/02/2016** | -0.01008545 | -0.009996007 | -0.01158 | -0.01648 |
| **01/03/2016** | 0.048281861 | 0.048337391 | 0.049434 | 0.051406 |
| **01/04/2016** | 0.03141563 | 0.031064089 | 0.029023 | 0.019789 |
| **01/05/2016** | 0.029874414 | 0.029771044 | 0.029683 | 0.028467 |
| **01/06/2016** | -0.007263024 | -0.007396891 | -0.0115 | -0.02011 |
| **01/07/2016** | 0.039621157 | 0.039640078 | 0.040851 | 0.044707 |
| **01/08/2016** | 0.000525721 | 0.00058763 | 0.003306 | 0.008583 |
| **01/09/2016** | 0.01756677 | 0.017502302 | 0.018487 | 0.016871 |
| **01/10/2016** | 0.002492256 | 0.002596255 | 0.001869 | 0.004009 |
| **01/11/2016** | -0.002315481 | -0.00238459 | -0.00725 | -0.01455 |
| **01/12/2016** | -0.020482418 | -0.020405868 | -0.01796 | -0.01232 |
| **mean** | **0.006824368** | **0.006815216** | **0.006402** | **0.005629** |
| **var** | **0.00076318** | **0.000759207** | **0.000782** | **0.000792** |
| **risk** | **0.02762572** | **0.027553704** | **0.027964** | **0.02814** |
| **return/risk** | **0.247029503** | **0.247343012** | **0.228921** | **0.20004** |

Once again, as was the case with Markowitz estimators, the realised sharpe ratios (return/risk) were significantly lower than the expected sharpe ratios. PCA 95 was the best performing sharpe ratio portfolio, with a ratio of 0.24734 as opposed to Markowitz’s 0.24702. As such, it will be taken forward for subsequent combination with Bayesian Inference.

Above is the evolution of the 1000 dollar investment visualized over 2016-17. This trend line is relatively similar between all portfolios with the exception of PCA 85, which seems to deviate and project lower post 05/2016.

## **6.1.4 Bayesian inferred weights for buy period ‘16-‘17**

Now, Bayesian Inference is examined for the buy period 2016-17. It is different in approach as it uses the 5 year period 2010-2015 beginning to represent prior beliefs, and then the entirety of 2015 as the observed to infer new estimators based on updated beliefs.

Thus, the standard estimators for 2010-2015 are examined:

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Parameter** | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| mean | 0.017149 | 0.010252 | 0.026419 | 0.018542 | 0.021395 | 0.00746 | 0.01276 | 0.024005 | 0.01157 | 0.012352 |
| cum var | 0.011457 | 0.019086 | 0.017583 | 0.019155 | 0.013095 | 0.004846 | 0.020259 | 0.02062 | 0.036888 | 0.034018 |
| cum risk | 0.107036 | 0.138151 | 0.132601 | 0.138403 | 0.114435 | 0.069614 | 0.142333 | 0.143597 | 0.192063 | 0.184439 |
| return/risk | 0.160214 | 0.07421 | 0.19924 | 0.133973 | 0.186962 | 0.107161 | 0.08965 | 0.167171 | 0.060239 | 0.066971 |

And the Observed data from 2016-17:

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Parameter** | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| mean | 0.011216 | 0.019459 | -0.00216 | 0.070607 | 0.015601 | 0.020595 | 6.47E-05 | -0.00015 | -0.01426 | -0.00132 |
| cum var | 0.015398 | 0.042469 | 0.020078 | 0.035687 | 0.02832 | 0.021553 | 0.019716 | 0.032835 | 0.027677 | 0.033115 |
| cum risk | 0.124087 | 0.20608 | 0.141699 | 0.18891 | 0.168287 | 0.146811 | 0.140414 | 0.181203 | 0.166363 | 0.181976 |
| return/risk | 0.090384 | 0.094427 | -0.01527 | 0.373759 | 0.092702 | 0.14028 | 0.000461 | -0.00081 | -0.08571 | -0.00724 |

Notice in the Observed data, APPL, CMCSA, MS and C log negative monthly expected returns, whilst AMZN mean monthly return balloon from 0.018542 to 0.070607.

Though the above plot all show all stock prices increasing from the start point in 01/15 to the end point in 12/15, the middle period from 04/2015 to 08/2015 experienced many drops. This is further corrobated upon examination of sheet 2 in excel main.xlsx in the github.

2015 was a very tumultuous year for the NYSE market, driven by multiple factors. From the guardian, the Greece debt default seemed to be a major driving factor, requiring US government bailout [10].

Nevertheless, it would be interesting to see how given this new information the posterior predictors derivation.

Recall from section 3.4.2:

Or:

Where and are inferred from 2010-2015 prior distribution and observed distribution .

Considering such, the following estimators where inferred:

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Parameter** | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| mean | 0.008756 | 0.012872 | -0.00093 | 0.062665 | 0.011927 | 0.017099 | -9.09E-04 | -0.00121 | -0.01385 | -0.00251 |
| cum var | 0.000982 | 0.002601 | 0.001386 | 0.002269 | 0.001847 | 0.001314 | 0.001333 | 0.002155 | 0.001938 | 0.002244 |
| cum risk | 0.031344 | 0.050995 | 0.037235 | 0.04763 | 0.042971 | 0.036248 | 0.036508 | 0.046419 | 0.044028 | 0.047366 |
| return/risk | 0.27935 | 0.252422 | -0.02488 | 1.315668 | 0.277568 | 0.471733 | -0.0249 | -0.0261 | -0.31456 | -0.05307 |

This was a very curious result, as it heavily favours AMZN for investment. The belief of return for AMZN was 0.062665, dwarfing the next best LMT at 0.008756. the return/risk was massive for AMZN, projecting 1.31 monthly returns. Predictably, the weightage outputted was as follows.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| Bayesian | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |

Everything in AMZN. However, this goes against the diversification principle that Markowitz put forth. If AMZN did crash, then the entire investment would be lost. The cumulative risk also projected is still relatively high at 0.04763 compared to LMT’s 0.031344. This is potentially a limitation of using the sharpe ratio to select a portfolio. In such cases, where extreme portfolio weights are skewed due to estimators predicting massive returns for a particular asset, it might require investor intervention to spread the budget across the assets and not just chance it on one.

This is probably due to the implementation of the posterior estimators. Notice in the formulas above, the observed covariance matrix and mean returns are scaled heavily by the number of oberservations n, which implies that the posterior distribution will be skewed very heavily towards the observed estimators. Implementing a scaling factor on n observations may be beneficial, and changeable based on whether the most recent trends in the observed data hold true.

Still, in the interest of finding out the evolution of the budget with this weightage, the returns for 2016-17 were found via excel:

|  |  |  |
| --- | --- | --- |
| Price | Marko | Bayesian |
| 31/12/2015 | 1000 | 1000 |
| **01/01/2016** | 952.2609792 | 868.4845681 |
| **01/02/2016** | 942.6569987 | 817.4703172 |
| **01/03/2016** | 988.1702331 | 878.3086229 |
| **01/04/2016** | 1019.214223 | 975.8837075 |
| **01/05/2016** | 1049.66265 | 1069.390053 |
| **01/06/2016** | 1042.038925 | 1058.781758 |
| **01/07/2016** | 1083.325713 | 1122.682692 |
| **01/08/2016** | 1083.89524 | 1137.995895 |
| **01/09/2016** | 1102.935778 | 1238.825947 |
| **01/10/2016** | 1105.684577 | 1168.563031 |
| **01/11/2016** | 1103.124385 | 1110.491418 |
| **01/12/2016** | 1080.529731 | 1109.455746 |

From the above table, the portfolio performed much better, realising 109.46 dollars in profit over Markowitz’s 80.53. the return of the year for the Bayesian portfolio was 10.95% compared to 8.05% achieved by the Markowitz portfolio. However, hindsight is also 20/20, and upon comparing the month on month returns however, a different picture was painted:

|  |  |  |
| --- | --- | --- |
| Returns | Marko | Bayesian |
| **01/01/2016** | -0.04774 | -0.13152 |
| **01/02/2016** | -0.01009 | -0.05874 |
| **01/03/2016** | 0.048282 | 0.074423 |
| **01/04/2016** | 0.031416 | 0.111094 |
| **01/05/2016** | 0.029874 | 0.095817 |
| **01/06/2016** | -0.00726 | -0.00992 |
| **01/07/2016** | 0.039621 | 0.060353 |
| **01/08/2016** | 0.000526 | 0.01364 |
| **01/09/2016** | 0.017567 | 0.088603 |
| **01/10/2016** | 0.002492 | -0.05672 |
| **01/11/2016** | -0.00232 | -0.04969 |
| **01/12/2016** | -0.02048 | -0.00093 |
| **mean** | **0.006824** | **0.011368** |
| **var** | **0.000763** | **0.005793** |
| **risk** | **0.027626** | **0.076111** |
| **return/risk** | **0.24703** | **0.149354** |

Though the monthly return was significantly higher in the Bayesian Portfolio at 0.011368 compared to Markowitz’s 0.006824, the risk experienced was also incredibly high at 0.076111 compared to Markowitz’s 0.024703. This shows that the realised Sharpe ratio for the Bayesian portfolio was lower than Markowitz’s, at 0.14935 to 0.24703 respectively.

This is better visualized by tracking the portfolio valuation through 2016-17:

Here, the variance experienced by the Bayesian portfolio was extremely significant, dropping heavily initially due to an initial -0.131512 loss experienced by the portfolio, and subsequently AMZN. Perspective is very important here. An investor who views the performance year on year on this portfolio would be very pleased, after all it achieved a 10.95% return on investment. But an investor who is actively monitoring his portfolio may be dissuaded after viewing the first few months 2016, especially considering the 13% loss experienced in Jan 2016.

PCA could prove useful here. By filtering the prior data, a more robust and trendlike distribution could achieved, thus reducing the inherent projected risk/return put forth by the posterior distribution estimator. This is explored in the next section.

## **8.1.5 PCA95-Bayesian inferred weights for buy period ‘16-‘17**

Recall that in section 6.1.3, it was decided PCA at a threshold of 95% cumulative variance would be used. Considering such, the denoised 2010-2015 estimators were retrieved:

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Parameter** | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| mean | 0.017149 | 0.010252 | 0.026419 | 0.018542 | 0.021395 | 0.00746 | 1.28E-02 | 0.024005 | 0.01157 | 0.012352 |
| cum var | 0.011455 | 0.019088 | 0.01757 | 0.019159 | 0.013094 | 0.004847 | 0.020265 | 0.020608 | 0.036779 | 0.034123 |
| cum risk | 0.107029 | 0.138159 | 0.132553 | 0.138416 | 0.114428 | 0.06962 | 0.142355 | 0.143553 | 0.191778 | 0.184723 |
| return/risk | 0.160224 | 0.074206 | 0.199312 | 0.133961 | 0.186973 | 0.107152 | 0.089636 | 0.167221 | 0.060329 | 0.066868 |

Compared to the standard 2010-2015 estimators:

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Parameter** | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| mean | 0.017149 | 0.010252 | 0.026419 | 0.018542 | 0.021395 | 0.00746 | 0.01276 | 0.024005 | 0.01157 | 0.012352 |
| cum var | 0.011457 | 0.019086 | 0.017583 | 0.019155 | 0.013095 | 0.004846 | 0.020259 | 0.02062 | 0.036888 | 0.034018 |
| cum risk | 0.107036 | 0.138151 | 0.132601 | 0.138403 | 0.114435 | 0.069614 | 0.142333 | 0.143597 | 0.192063 | 0.184439 |
| return/risk | 0.160214 | 0.07421 | 0.19924 | 0.133973 | 0.186962 | 0.107161 | 0.08965 | 0.167171 | 0.060239 | 0.066971 |

The observed data was not subject to PCA in anticipation of potential loss of crucial trends experienced in 2015. Thus, it was the same as the observed data table in section 6.1.4.

The estimators inferred given the denoised prior data were as follows:

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Parameter** | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| mean | -0.00241 | -0.00659 | 0.019409 | 0.053528 | 0.000366 | 0.009521 | -6.10E-03 | -0.00452 | -0.01636 | -0.0188 |
| cum var | 0.000379 | 0.001307 | 0.0011 | 0.001378 | 0.001191 | 0.000555 | 0.001055 | 0.001285 | 0.001761 | 0.001706 |
| cum risk | 0.01948 | 0.036153 | 0.033171 | 0.037115 | 0.034512 | 0.023553 | 0.032476 | 0.035851 | 0.041965 | 0.0413 |
| return/risk | -0.12377 | -0.18233 | 0.585122 | 1.44221 | 0.010611 | 0.404261 | -0.18794 | -0.12598 | -0.38979 | -0.45517 |

Recall the posterior estimators without PCA:

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **Parameter** | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| mean | 0.008756 | 0.012872 | -0.00093 | 0.062665 | 0.011927 | 0.017099 | -9.09E-04 | -0.00121 | -0.01385 | -0.00251 |
| cum var | 0.000982 | 0.002601 | 0.001386 | 0.002269 | 0.001847 | 0.001314 | 0.001333 | 0.002155 | 0.001938 | 0.002244 |
| cum risk | 0.031344 | 0.050995 | 0.037235 | 0.04763 | 0.042971 | 0.036248 | 0.036508 | 0.046419 | 0.044028 | 0.047366 |
| return/risk | 0.27935 | 0.252422 | -0.02488 | 1.315668 | 0.277568 | 0.471733 | -0.0249 | -0.0261 | -0.31456 | -0.05307 |

This is alarming, as APPL has suddenly become a potentially worthwhile investment, going from a -0.02488 predicted return/risk rate to a dramatic 0.585122. The difference between the risk/return rate for the 2010-2015 denoised and standard data was very minor: 0.199312 and 0.19924.

This result was very anomalous, and could not be explained mathematically. It was concluded that the error was technical, and that MATLAB struggled to take the inverse of the denoised data covariance matrix, resulting in an inaccurate posterior covariance matrix, and subsequently, an inaccurate posterior expected returns vector.

Predictably, the weights assigned given the posterior estimators above provided some weight to APPL:

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | **LMT** | **MSFT** | **APPL** | **AMZN** | **V** | **MCD** | **WFC** | **CMCSA** | **MS** | **C** |
| Bayesian PCA 95 | 0 | 0 | 0.028 | 0.6947 | 0 | 0.2765 | 0 | 0 | 0 | 0 |

The results, though faulty, still provide some interesting insights:

|  |  |  |  |
| --- | --- | --- | --- |
| Price | Marko | Bayesian | Bayesian 95 |
| 31/12/2015 | 1000 | 1000 | 999.2 |
| **01/01/2016** | 952.2609792 | 868.4845681 | 918.9295459 |
| **01/02/2016** | 942.6569987 | 817.4703172 | 867.8935267 |
| **01/03/2016** | 988.1702331 | 878.3086229 | 933.3001547 |
| **01/04/2016** | 1019.214223 | 975.8837075 | 998.924699 |
| **01/05/2016** | 1049.66265 | 1069.390053 | 1055.143362 |
| **01/06/2016** | 1042.038925 | 1058.781758 | 1042.615018 |
| **01/07/2016** | 1083.325713 | 1122.682692 | 1083.001547 |
| **01/08/2016** | 1083.89524 | 1137.995895 | 1089.484906 |
| **01/09/2016** | 1102.935778 | 1238.825947 | 1160.678159 |
| **01/10/2016** | 1105.684577 | 1168.563031 | 1105.467014 |
| **01/11/2016** | 1103.124385 | 1110.491418 | 1080.002284 |
| **01/12/2016** | 1080.529731 | 1109.455746 | 1086.426743 |

Despite incorrectly predicting APPL (and MCD) to be worthwhile investments, the PCA95 Bayesian Portfolio still had a higher return, achieving 5.13 dollars more than Markowitz. However, this cannot be attributed to more accurate and robust estimators, as the estimators themselves are anomalous and unexplainable.

Still, examining the returns results proves that diversification is important when it comes to risk mitigation:

|  |  |  |  |
| --- | --- | --- | --- |
| Returns | Marko | Bayesian | Bayesian 95 |
| **01/01/2016** | -0.04774 | -0.13152 | -0.08033 |
| **01/02/2016** | -0.01009 | -0.05874 | -0.05554 |
| **01/03/2016** | 0.048282 | 0.074423 | 0.075363 |
| **01/04/2016** | 0.031416 | 0.111094 | 0.070315 |
| **01/05/2016** | 0.029874 | 0.095817 | 0.056279 |
| **01/06/2016** | -0.00726 | -0.00992 | -0.01187 |
| **01/07/2016** | 0.039621 | 0.060353 | 0.038736 |
| **01/08/2016** | 0.000526 | 0.01364 | 0.005986 |
| **01/09/2016** | 0.017567 | 0.088603 | 0.065346 |
| **01/10/2016** | 0.002492 | -0.05672 | -0.04757 |
| **01/11/2016** | -0.00232 | -0.04969 | -0.02304 |
| **01/12/2016** | -0.02048 | -0.00093 | 0.005949 |
| **mean** | **0.006824** | **0.011368** | **0.008302** |
| **var** | **0.000763** | **0.005793** | **0.002843** |
| **risk** | **0.027626** | **0.076111** | **0.053324** |
| **return/risk** | **0.24703** | **0.149354** | **0.155687** |

Compared to the Bayesian Portfolio, the risk was much lower for the PCA95 Bayesian Portfolio at 0.053324 compared to the formers 0.076111. This asserts that spreading out an investment over various assets (even if they don’t perform well) will mitigate the volatility of the portfolio.

This is further reinforced by visualising the portfolio valuation through 2016:

Here, the Dark green line is the faulty PCA 95 Bayesian Model. It constantly lives within the bounds of the Bayesian Portfolio, which consisted of the AMZN asset. This is because while the portfolio consisted of 0.6947 asset weight AMZN, the remaining 0.3153 weightage was spread across other assets that were not as risky as AMZN.

Unfortunately, as the PCA 95 Bayesian Portfolio was faulty, it was omitted from the dynamic investment scenario. However, from experiment 1 and the analysis above, the theoretical benefits are better risk predictors, and subsequently, more diversification in generated portfolios.

## **9.1 Evaluations and Conclusions:**

At a glance, each of the three functions performed their function well. All the created frameworks, except the BayesianPCA.mlx, performed well under the dynamic scenario. They are easy to use and only require an excel sheet of stocks and their stock prices. Each framework can handle any amount of assets, and any amount of stock prices, and output returns, expected and the only thing a potential investor using these frameworks would have to do Is provide data via Excel, and choose his parameters for portfolio generation.

The Markowitz Portfolio generator was consistent throughout testing, providing logical portfolio weights based on a selected range of historical data. It also provided the highest average monthly returns in Experiment 1’s dynamic scenario. It is very useable for a potential investor looking for a standard portfolio generation technique based on historical data, and proved to be robust enough to handle changing historical data.

The PCA Mean Variance Portfolio Generator took this one step further and proved to be the most robust, achieving higher portfolio diversification than both Markowitz and Bayesian frameworks. It also provided the most stable returns with the lowest volatility and subsequently, the highest Sharpe ratio out of sample. It is also superbly easy to use, requiring only an excel sheet of any dimensions to be inputted, and a noise reduction threshold to be selected. An experienced Financial Manager could greatly benefit from onlyPCA1.mlx as it would handle noise reduction for him. For BP2 however, decreasing the noise threshold proved to be futile, resulting in lower returns and higher risk experienced, but the project author lacks the financial vintage to make decisions on what constitutes as noise, and what doesn’t.

The Bayesian framework presents itself as a mixed bag. But also, no conclusions can be drawn regarding it as it was severely limited by a number of factors:

* 1. By assuming that all returns in all the data ranges are normal, the posterior distrubutions were derived in a way that may not have been correct for the data.
  2. Scaling the posterior estimators by n samples proved to be detrimental, as the framework put forth estimators that more closely replicated the observed data trend with less important on the prior information. A scaling factor could be greatly beneficial here.
  3. Due to the above effect, as evidenced in Experiment 2, the framework was inhibited by the short selling rule. This resulted in the Bayesian framework having to allocate its weight to the few assets that did not indicate a negative return, resulting in very undiverse portfolios.

In the dynamic scenario, due to the above limitations, it proved to make huge bets that more often than not, did not pay off, resulting in the lowest mean monthly returns, and highest risk, resulting in the worst sharpe ratio out of sample.

Also of note, despite the statements made above, was that the testing procedure undertaken by the project author was suboptimal to explore and evaluate the performance and functionality of each framework. The analysis and testing was far too focused on how the methods might have translated to more accurate predictions, and by extension, a number of features were not tested. Namely, the shortselltoggle feature, and the ability to be dynamic to any number of assets and stock prices.

Furthermore, there were many sources of inaccuracy in the testing process. For one, the target\_returns function in the quadprog solver only put forth 100 target returns, and so the sharpe portfolio presented may not have actually represented the highest expected/return ratio.

Furthermore, testing the performance of the weights via excel introduced many decimal errors. Whilst it is true that the values used were in the thousands, recall that the difference in profit between Markowitz and PCA 95 was only 3 dollars, or 0.3% of the initial investment of 1000$, very much within the bounds of error induced by excel calculations.

As a result, and very unfortunately, the report can not conclude but only reiterate the promise of all the frameworks generated due to poor testing procedure.

## **10 Further Work:**

Given the above conclusions, or lack thereof, there are still a multitude of ways conclusions could be derived. For one, the promise of each framework is dynamic allocation of weights irrespective of asset stock price is already implemented, but should be tested with different data sets instead of just one as this project report did.

Additionally, the framework as it stands only outputs the highest sharpe ratio portfolio, but with a very simple implementation of a scale factor k to the risk in sharpe ratio formula:

The framework can be tuned to output either risk-averse portfolios by setting k very high, for example 1000, or be tuned to output return oriented portfolios by setting k = 0.01 (not 0.)

Furthermore, the framework can theoretically handle short selling. This should be tested at length with an experienced financial manager, who can determine whether short sell positions are viable or not.

Also, the only manual and perhaps tedious input to each framework is the excel sheet. This can be automated via APIs. AlphaVantage [11] offers a free API service that can be plugged into MATLAB. Via development of an API, an investor would not be required to collate stock price information himself for evaluation.

The promise of the BayesianPCA.mlx framework was immense, offering the observed benefit of PCA in the experiment combined with the potential offered by Bayesian inference. This is something the project author is actively working on.

# Lastly, Novel approaches to robust portfolio optimization include uncertainty bounds; Uncertainty bounds in portfolio optimization are used to account for the inherent uncertainty in estimating asset returns, volatilities, and correlations. These bounds provide a range within which the true values are expected to lie, allowing for more robust decision-making. This was actually a project goal, but was not implemented due to poor time management. Appendix section C introduces a few types of uncertainty bounds that could be implemented in addition to PCA and Bayesian inference.

# References

|  |  |
| --- | --- |
| [1] | C. M. Cap, “BRK-B Market Capitalization,” [Online]. Available: https://companiesmarketcap.com/berkshire-hathaway/marketcap/. |
| [2] | A. Hayes, “Investopedia,” 2023. [Online]. Available: https://www.investopedia.com/terms/f/financial-market.asp. |
| [3] | K. Axelton, “Experianhttps://www.experian.com/blogs/ask-experian/what-is-financial-portfolio/,” 2022. [Online]. |
| [4] | IBM, “Monte Carlo Simulation,” 2017. [Online]. Available: https://www.ibm.com/topics/monte-carlo-simulation. |
| [5] | H. Markowitz, “Portfolio Selection,” 1952. [Online]. Available: https://www.math.hkust.edu.hk/~maykwok/courses/ma362/07F/markowitz\_JF.pdf. |
| [6] | S. S. A. Maiyahi, “Eficient Frontier Curve,” [Online]. Available: https://www.linkedin.com/pulse/efficient-frontier-curve-sultan-saif-al-maiyahi/. |
| [7] | Investopedia, “risk free asset definitions,” [Online]. Available: https://www.investopedia.com/terms/r/riskfreeasset.asp#:~:text=key%20takeaways,compensated%20for%20taking%20a%20chance.. |
| [8] | U. o. Pennsylvannia, “Proof of Bayes Theorem,” [Online]. Available: https://www.hep.upenn.edu/~johnda/Papers/Bayes.pdf. |
| [9] | Y. Finance, “Histroical Stock Prices,” [Online]. Available: These ten stocks were chosen as they branch from infrastructure (such as amazon and ford) to Healthcare (Pfizer and Moderna) to Technology (such as Microsoft and Apple) to hospitality (Mcdonalds, Starbucks) and Social Media (Meta and Netflix.) Having a wi. |
| [10] | T. Guardian, “Five factors that shook the world’s markets in 2015,” [Online]. Available: https://www.theguardian.com/business/2015/dec/27/five-factors-shocked-stock-markets-2015. |
| [11] | A. Vantage, “Alpha Vantage,” [Online]. Available: https://www.alphavantage.co. |
| [12] | N. L. F. Medicine, “The origins of Covid 19, a brief overview,” [Online]. Available: https://www.ncbi.nlm.nih.gov/pmc/articles/PMC9874793/#:~:text=The%20most%20widespread%20coronavirus%20to,disease%20(COVID‐19).. |
| [13] | Forbes, “The aftermath and impact of covid-19 on stock markets,” [Online]. Available: https://www.forbes.com/sites/theyec/2023/02/10/the-aftermath-and-impact-of-covid-19-on-stock-markets/. |
| [14] | BBC, “Coronavirus: How the pandemic has changed the world economy,” [Online]. Available: https://www.bbc.co.uk/news/business-51706225. |
| [15] | N. Hoang, “PORTFOLIO OPTIMIZATION METHODS: THE MEAN-VARIANCE APPROACH,” 2019. [Online]. Available: https://egrove.olemiss.edu/cgi/viewcontent.cgi?article=2060&context=hon\_thesis. |

## **11 Appendix:**

## **A: Github Repo**

https://github.com/pv319/FYP

## **B: Mathematical Background: Analytical Solving of Minimization Problem**

This section evaluates how the minimization problem stated can be solved via quadratic algebra and Lagrange Multipliers.

Consider the minimization problem:

Subject to:

Where represents the vector of asset weights, represents the covariance of returns, or risk in this constant, a vector of ones in the same dimensions as , representing a vector of expected returns for each asset and representing a target portfolio return.

This problem can be solved by introducing Lagrange Multipliers for the constraints:

And so, considering the new Lagrange multipliers, A Langrangian function can be created:

This can now be differentiated with regard to the three unknowns: and the resulting partial differential equations set to 0:

Derivative with respect to :

Derivative with respect to

Derivative with respect to

These equation represent a system of linear equations:

Representing in terms of from the first equation in the system:

Substituting the new into the second equation in the system:

(a)

And simplifying:

(b)

Substituting the new into the third equation in the system:

And simplifying:

(c)

Let , , , equations (b) and (c) become:

Solving for :

Substituting new values for back into equation (a) for :

Thus, optimal portfolio weights for can be found to minimize risk while achieving a portfolio return given .

**D: Different Types of Uncertainty that could be implemented:**

**Box Uncertainty:**

A box uncertainty refers to a specific type of representation of uncertainty where instead of having a single probability distribution, a range of possible distributions is encapsulated within a "box" or a bounding region. This indicates that the true distribution of a parameter or variable is uncertain within certain bounds. In the projects case, the ‘true’ expected return will be within the box parameters, thus allowing for plus/minus estimation error.

For this project, Box uncertainty could be defined like so:

Here, the Uncertainty classifies the expected return matrix with an estimator error of .

**Circular Uncertainty:**

A circular uncertainty set is a concept often used in optimization and decision-making under uncertainty. In this context, an uncertainty set represents a range of possible values or scenarios for uncertain parameters that affect the outcome of a decision or optimization problem.

In a circular uncertainty set, the uncertainty is modelled as a circle in a two-dimensional space. This circle represents all possible values or scenarios for the uncertain parameter(s) that are considered equally likely or plausible. Each point within the circle corresponds to a specific scenario or value that the uncertain parameter(s) could take.

The circular shape implies that there is equal uncertainty in all directions or dimensions within the set. This means that any point within the circle is considered equally probable or plausible given the available information about the uncertainty.

Circular uncertainty sets are particularly useful when the uncertainty in the problem can be adequately characterized by a single parameter or when the interactions between uncertain parameters are relatively simple. They provide a geometrically intuitive way to represent uncertainty and can be used in various optimization and decision-making algorithms to account for uncertainty in the problem formulation.

For this project, the Circular Uncertainty set could be defined as follows:

Where could be obtained experimentally by plotting the stocks price against time, and finding the max estimation error from the expected return denoted by .

Ellipsoidal Uncertainty:

Ellipsoidal uncertainty is another representation of uncertainty commonly used in statistics and data analysis, particularly in the context of multivariate data. Instead of bounding uncertainty with a box-like shape, ellipsoidal uncertainty represents uncertainty using ellipsoids.

An ellipsoid is a geometric shape that resembles a stretched or compressed sphere. In the context of uncertainty, an ellipsoid represents a region in which the true value of a parameter or variables is likely to lie, with the centre of the ellipsoid typically representing the most probable value.

The shape and orientation of the ellipsoid reflect the covariance structure of the uncertainty. In other words, it captures how different variables or parameters are related to each other and how uncertainty in one variable might affect uncertainty in another.

Ellipsoidal uncertainty is particularly useful when dealing with multivariate data because it allows for a more nuanced representation of uncertainty compared to simple intervals or boxes. By visualizing uncertainty as an ellipsoid, financial managers can understand not only the magnitude of uncertainty but also how different variables are correlated or interact with each other. This would be particularly useful in the projects case, as it considers the covariance matrix, which is surrogated as the risk for the projects analysis.

For this project, it could be represented as follows:

Where refers to the inverse of the global covariance matrix and refers to the standard deviation for stock price in historical data.

## **C: Ethics, Legal and Safety Action**

Fortunately, as the project is physical in no regard, there are no Safety procedures necessary to follow whilst the project was being undertaken.

For Ethics and Legality, the project only utilized publicly available trading information available on yahoo finance, and care was taken to reference all work referenced and cited.

Furthermore, a fully licensed issue of MATLAB software as well Microsoft Excel was utilized.

## 